8 Proportional reasoning

Mastery Professional Development

8.2 Understanding graphical representations of proportionality

Guidance document | Key Stage 4

Click the heading to move to that page.

Please note that these materials are principally for professional development purposes. Unlike a textbook scheme they are not designed to be directly lifted and used as teaching materials. The materials can support teachers to develop their subject and pedagogical knowledge and so help to improve mathematics teaching in combination with other high-quality resources, such as textbooks.

Making connections

Building on the Key Stage 3 mastery professional development materials, the NCETM has identified a set of five 'mathematical themes' within Key Stage 4 mathematics that bring together a group of 'core concepts'.

The second of the Key Stage 4 themes (the eighth of the themes in the suite of Secondary Mastery Materials) is *Proportional reasoning*, which covers the following interconnected core concepts:

8.1 Working with direct and inverse proportion

8.2 Understanding graphical representations of proportionality

This guidance document breaks down core concept *8.2 Understanding graphical representations of proportionality* into three statements of **knowledge, skills and understanding**:

8.2 Understanding graphical representations of proportionality

- 8.2.1 Solve growth problems including compound interest
- 8.2.2 Interpret the gradient as a rate of change
- 8.2.3 Interpret the gradient at a point on a curve as the instantaneous rate of change

Then, for each of these statements of knowledge, skills and understanding we offer a set of **key ideas** to help guide teacher planning:

- 8.2.1 Solve growth problems including compound interest
  - 8.2.1.1 Understand that a percentage change can be calculated using a single multiplier
  - 8.2.1.2 Understand that a repeated percentage change can be calculated through repeated use of a single multiplier
  - 8.2.1.3 Represent growth problems graphically
  - 8.2.1.4 Use and apply exponentiation to solve growth problems
- 8.2.2 Interpret the gradient as a rate of change
  - 8.2.2.1 Understand that a rate of change is a multiplicative comparison between two quantities
  - 8.2.2.2 Make connections between linear graphs and multiplicative relationships
  - 8.2.2.3 Know how the key features of a proportional relationship are represented graphically
  - 8.2.2.4 Interpret the gradient as a rate of change
  - 8.2.2.5 Solve problems with rates of change represented algebraically and graphically
- 8.2.3 Interpret the gradient at a point on a curve as the instantaneous rate of change

8.2.3.1 Understand that the gradient of a continuous curve constantly changes

8.2.3.2 Understand that a tangent to a curve is a unique straight line that meets the curve at exactly one point

8.2.3.3 Interpret the gradient of the tangent to a curve as the rate of change of the relationship represented by the curve at that instant

Overview

This core concept is about multiplicative structure, particularly recognising and describing the characteristics of a multiplicative relationship from its graphical representation. Central to this is an understanding of the linear graph of direct proportionality.

At Key Stage 3, students explored the connections between linear equations and their corresponding graphs. They learned about the two significant features of a straight line that make it unique in the plane and described these as 'gradient' and 'y-intercept'. The extent to which students understand how these describe the rate at which y changes with respect to x and the position of the line in the plane may vary, which will have a bearing on how much this content needs to be reviewed as part of Key Stage 4 teaching.

At Key Stage 4, the graphical representation of a directly-proportional relationship is explored and students' understanding of the gradient of a linear function as a rate of change is developed. Recognising that two quantities that are in direct linear proportion can be represented by a straight-line graph passing through the origin is an important part of students' developing understanding of multiplicative structure. Recognising how the gradient of a straight-line graph relates to the functional multiplier or constant of proportionality is indicative of an understanding of the multiplicative structure that underpins proportional relationships. Understanding gradient as a rate of change (change in the value of y divided by change in the value of x) is key to students recognising why the constant of proportionality is equivalent to the gradient for directly-proportional linear relationships.

Students will have been introduced to inverse proportion at Key Stage 3 and have some experience of reciprocal graphs. Understanding why the graph of y against x for an inversely-proportional relationship such as $y = \frac{2}{x}$ is a curve and not a straight-line is important in recognising how the two relationships differ, and how the constant of proportionality for inverse proportion is the product of corresponding values of x and y rather than the quotient. Students should recognise that with an inversely-proportional relationship between y and x the gradient is not constant.

Contrasting the graphical representation of a directly-proportional relationship as a straight line passing through the origin with the graph of $y = \frac{2}{x}$, which never touches the axes, provides insight into the multiplication and division properties of zero and highlights the multiplicative structure of proportional relationships. Although the graphs of directly and inversely proportional relationships between x and y differ, the underlying multiplicative structure remains.

If we consider the relationship $y = \frac{2}{x}$ as $y = 2 \times \frac{1}{x}$ we could choose to plot the graph of y against $\frac{1}{x}$ and we would then get a linear graph with gradient 2. This reveals how an inversely proportional relationship between y and x is the same as a directly proportional relationship between y and $\frac{1}{x}$. It is unlikely that we would want to describe the relationship in this way to students when first exploring graphical representations, however, thinking more deeply about this relationship provides insight into the multiplicative relationship underpinning proportionality.

Students may have met exponential graphs at Key Stage 3. If so, it is likely that they will have worked with graphs of a variety of functions, including those with an exponential relationship, to find approximate solutions to problems. At Key Stage 4, compound interest provides a context within which to explore exponential graphs further. For simple interest the amount of interest earned is directly proportional to the time invested for, so would be represented as a straight-line graph starting at (0, 0). The relationship between the total amount including interest and the number of years invested for is represented by a linear graph that does not start at the origin, as the total amount at the start (when $t = 0$) is equal to the original investment amount. Compound interest is not directly proportional to time. Plotting the total amount including interest against time for compound interest gives a curve and demonstrates that the total over time is a non-linear function that increases exponentially.

Prior learning

Multiplicative relationships underpin many aspects of mathematics at Key Stage 3. As students' understanding develops, the underlying mathematical structure of proportionality is explored. Students' understanding of the connection between multiplicative relationships and direct proportion is developed to recognise when a relationship is proportional, and establish that a unifying structure connecting different types of proportionality exists.

At Key Stage 3, students developed their view of multiplication from one focused on repeated addition to that of a multiplicative relationship and began to recognise that it is possible to go from one number to another by multiplying as well as thinking additively. However, when working with percentages it is likely that students will still be used to an additive approach, and so to increase an amount by 20 per cent, for example, a common approach may be to find 20 per cent and add it on (often finding 10 per cent first by dividing by 10 and doubling to get 20 per cent). Whilst this produces a correct solution, it is not only a less efficient strategy than using a single multiplier, but also suggests a limited understanding of multiplicative structure.

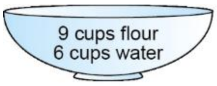
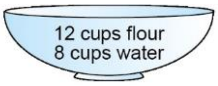
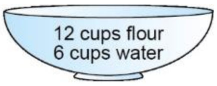
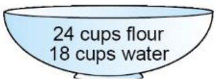
At Key Stage 3, students learnt to express relationships between variables graphically as well as algebraically. Significant attention was given to exploring linear relationships and their representation as straight-line graphs. The recognition that all linear relationships have certain key characteristics can be built on to establish how the key features of a proportional relationship are represented graphically. For example, students' understanding of gradient at Key Stage 3 is often linked to an association with slope in the graphical representation of a linear equation and recognition that as the value of the gradient increases, the graph becomes steeper. The significance of the slope of a line, however, is not always fully appreciated. Students often do not recognise that the slope of a line tells us how something changes over time, and that if we can establish the slope (gradient) we can understand the rate of change.

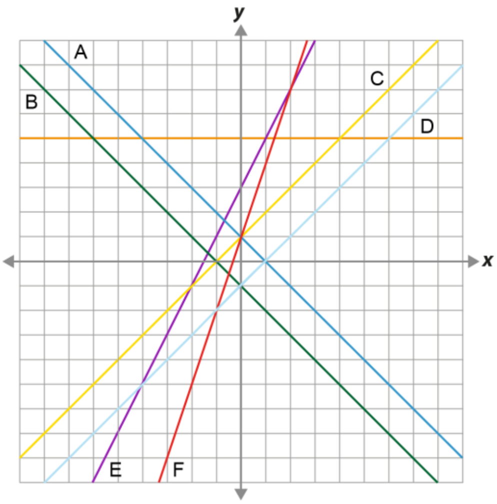
The core concept documents '*3.1 Understanding multiplicative relationships'* and '*4.2 Graphical representations'* from the Key Stage 3 PD materials both explore in more depth the prior knowledge required for this core concept.

Checking prior learning

The following activities from the NCETM secondary assessment materials, Checkpoints and/or Key Stage 3 PD materials offer a sample of useful ideas for assessment, which you can use in your classes to check understanding of prior learning.

Reference	Activity
Checkpoints '*Multiplicative relationships*', 'Checkpoint 18: Shades of orange'	This orange paint is made using two cans of yellow for every one can of red.  Graham, Phil and Alison are given the orange paint. They each add three more cans to the mix, using only red and yellow. What can you say about the colour of the cans of paint that were added in each case, below? a) Graham's paint turns lighter. 

	b) Phil’s paint turns the same colour.  c) Alison’s paint turns darker.  One can of red and one can of yellow is added to each of the three paint mixes. Does each one become more red or more yellow?
Checkpoints ‘*Multiplicative relationships*’, ‘Checkpoint 19: Dough’	Most bread doughs start with flour and water. To make chapati dough, for every two cups of flour you add one cup of water. To make pizza-base dough, for every three cups of flour, you add two cups of water. In each bowl, is it chapati dough, pizza base dough, or neither? a)  b)  c)  d)  e)  f)  What would you need to add to each bowl to swap the dough to make the other kind? Is there more than one way to do this?
Key Stage 3 PD materials document ‘4.2 *Graphical representations*’, Key idea 4.2.2.3, Example 2	Which of these straight lines pass through the origin? $y = 0x + 4$ $\quad$ $y = -2x$ $y = 3x + 0$ $\quad$ $3y + 5x = 0$ $2y = 4$

Key Stage 3 PD materials document '4.2 *Graphical representations*', Key idea 4.2.2.3, Example 7	Match these equations to the lines on the graph below. $y = 3x + 1$ $\quad$ $y = x - 1$ $y = 2x + 1$ $\quad$ $y = 1 - x$ $y = 2x + 3$ $\quad$ $y = 5$ $y = x + 1$  Which equation cannot be one of the lines? Explain how you know.

Key vocabulary

Key terms used in Key Stage 3 materials

- gradient
- linear

The NCETM's mathematics glossary for teachers in Key Stages 1 to 3 can be found here.

Key terms introduced in the Key Stage 4 materials

Term	Explanation
constant of proportionality	Where x and y are quantities in proportion, the constant of proportionality, $k = \frac{y}{x}$. The constant of proportionality can be a negative number as well as a positive number.
direct proportion	Two variables A and B are in direct proportion if the algebraic relation between them is of the form $A = kB$, where k is a constant; an alternative form of the equation is $\frac{A}{B} = k$. The graphical representation of this relationship is a straight line through the origin, and k is the gradient of the line. Commonly, A and B are replaced by y and x, but proportionality holds when $A = y$ and $B = x^n$; y is said to be directly proportional to x -squared (or cubed etc.)
instantaneous rate of change	The rate of change at any given point is called the instantaneous rate of change and can be estimated from non-linear relationships by drawing a tangent to the curve and calculating its gradient at that point. The rate of change for a curve is not constant but constantly changes.
inverse proportion	Two variables A and B are inversely proportional if the algebraic relation between them is of the form $AB = k$ where k is a constant; an alternative form of the equation is $A = \frac{k}{B}$. The relations are valid for $A \neq 0$ and $B \neq 0$. If A is inversely proportional to B then A is directly proportional to $\frac{1}{B}$. In the graphical representation of this function, the axes are asymptotes to the curve. Commonly, A and B are replaced by y and x, but inverse proportionality holds when $A = y$ and $B = x^n$; y is said to be inversely proportional to x -squared (or cubed etc.)
rate of change	A rate of change describes how one quantity changes in relation to another quantity. The gradient of a line represents the rate of change and is determined by the change in y divided by the change in x.
simple/compound interest	In savings (or loans) banks pay (or charge) interest on the amount invested (or borrowed). An interest rate is usually specified, and this is applied at specified periods, for example annually. The simple interest is what is added to the savings (loan) at the end of the specified period. Example: a saver invests £10 000 in a savings account that gives 1 per cent interest per year. At the end of the year the simple interest is 1 per cent of $£10000 = £10000 \times \frac{1}{100} = £100$. Usually, this is then added to the original £10 000, so that the amount now invested is £10 100. When interest is added over and over again in this way it is called compound interest.
tangent	A line is a tangent to a curve when it meets the curve in one and only one point and matches the curve's instantaneous gradient at that point. A tangent to a curve does not have to pass through the origin.

Key ideas

In the following list of the key ideas for this core concept, selected key ideas are marked with a . These key ideas are expanded and exemplified in the next section – click the links to be taken direct to the relevant exemplifications. Within these exemplifications, we explain some of the common difficulties and misconceptions, provide examples of possible pupil tasks and teaching approaches and offer prompts to support professional development and collaborative planning.

8.2.1 Solve growth problems including compound interest

The context of compound interest is helpful in moving students' thinking from an additive to a multiplicative approach. When calculating compound interest accumulation after just a few years, an additive approach does not appear too onerous. However, its limitations for growth over longer periods of time and its susceptibility to error support the need for a more efficient approach. A multiplier method provides such an approach, with exponentiation reducing the need for repeated percentage-change calculations. Students often assume that an increase of 20 per cent followed by an increase of 20 per cent is the same as an increase of 40 per cent. While this may be the case in the context of simple interest, it is not true for compound interest, where the amount gaining interest increases with each iteration. Attention should be drawn to this difference.

Compound interest is an example of exponential growth, and the value of an investment over time can be represented by an exponential curve. Students may have used graphical representations of exponential functions before to find approximate solutions to a problem. However, it's unlikely that they will have produced the representations for themselves. Comparing simple and compound interest problems graphically is a powerful way of demonstrating the benefits of compound returns over interest calculated on the original amount only.

- 8.2.1.1 Understand that a percentage change can be calculated using a single multiplier
- 8.2.1.2 Understand that a repeated percentage change can be calculated through repeated use of a single multiplier
- 8.2.1.3 Represent growth problems graphically
- 8.2.1.4 Use and apply exponentiation to solve growth problems

8.2.2 Interpret the gradient as a rate of change

Students need to understand that the gradient of a graph represents the rate of change between two variables, and that a linear graph that goes through the origin describes a proportional relationship. They may have some understanding of the connection between the algebraic equation of a linear function and its graphical representation, but may not yet have insight into the multiplicative relationship that it represents.

When considering non-linear relationships, such as quadratic functions of the form $y = kx^2$, students may describe the graphical representation of $y = 3x^2$ as having a 'steeper curve' than the function $y = 2x^2$. They may assume, based on their understanding of gradient as a measure of 'slope' or 'steepness', that the multiplier (k) between corresponding values of x and y describes the gradient. As the gradient of the curve is not constant, this provides a contradiction, but students might recognise that for a given value of x the gradient of one curve is greater than the other. A deeper understanding can be gained through plotting the graph of y against x^2 and attention being drawn to the gradient (k).

At Key Stage 4 the development of students' understanding of gradient as a measure of slope to a rate of change is important for a deep and connected understanding. Recognising that the functional multiplier is synonymous with the gradient of a linear proportional function, because gradient represents unit rate, is a key connection for students to make. The double number line provides a bridge between the multiplicative relationship and its graphical representation, and can be used to support this understanding, as well as providing a means to recognise the difference between the scalar and functional multipliers and build an understanding of multiplicative structure.

8.2.2.1 Understand that a rate of change is a multiplicative comparison between two quantities

8.2.2.2 Make connections between linear graphs and multiplicative relationships

8.2.2.3 Know how the key features of a proportional relationship are represented graphically

8.2.2.4 Interpret the gradient as a rate of change

8.2.2.5 Solve problems with rates of change represented algebraically and graphically

8.2.3 Interpret the gradient at a point on a curve as the instantaneous rate of change

At Key Stage 3, students' calculation of gradient will have been for linear functions only; gradients of curved graphs is a new concept at Key Stage 4. Students need to understand that a tangent is a line that touches a curve at a single point and matches the curve's slope at that point. Students will need practice at constructing a reasonable estimate of the tangent to a curve; asking them to visually balance the gaps either side of the point of contact between the line and the curve can help with this. Since students are constructing the tangent 'by eye', they should appreciate that any calculations related to the line drawn will be estimates.

Calculating the gradient of the line they have drawn requires the same mathematics as finding the gradient of a linear function. Giving students the opportunity to find the gradient of the tangent to a curve at multiple points will help them to recognise that the gradient for a curve is not constant. The gradient of the tangent to the curve at a particular point being different to the gradient of the tangent to the curve at another point exemplifies the constant versus constantly changing gradient of straight lines and curves.

For a linear function through the origin, the gradient gives the multiplier for the proportional relationship; with non-linear functions, the gradient of the tangent describes the rate of change between the two variables at that point, but is not related to the multiplier.

Exploring the gradient of curves that represent directly proportional relationships helps to develop students' understanding of the underlying multiplicative structure of proportionality. Examine a variety of inversely-proportional and non-proportional curves to provide insight into what the gradient of the tangent to a curve represents. Graphical representations of relationships involving different measures can provide a context for the rate of change as a rate of measure that has some meaning.

8.2.3.1 Understand that the gradient of a continuous curve constantly changes

8.2.3.2 Understand that a tangent to a curve is a unique straight line that meets the curve at exactly one point

8.2.3.3 Interpret the gradient of the tangent to a curve as the rate of change of the relationship represented by the curve at that instant

Exemplified key ideas

In this section, we exemplify the common difficulties and misconceptions that students might have and include elements of what teaching for mastery may look like. We provide examples of possible student tasks and teaching approaches (in italics in the left column), together with ideas and prompts to support professional development and collaborative planning (in the right column).

The thinking behind each example is made explicit, with particular attention drawn to:

Deepening	How this example might be used for **deepening** all students' understanding of the structure of the mathematics.
Language	Suggestions for how considered use of **language** can help students to understand the structure of the mathematics.
Representations	Suggestions for key **representation(s)** that support students in developing conceptual understanding as well as procedural fluency.
Variation	How **variation** in an example draws students' attention to the key ideas, helping them to appreciate the important mathematical structures and relationships.

In addition, questions and prompts that may be used to support a professional development session are included for some examples within each exemplified key idea.

These are indicated by this symbol.

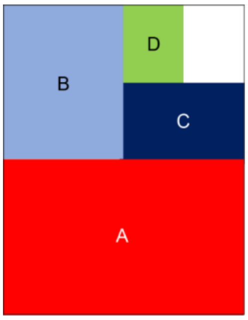
8.2.1.2 Understand that a repeated percentage change can be calculated through repeated use of a single multiplier

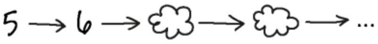
Common difficulties and misconceptions

It is common for students to work additively when working with percentage change: they will find 20 per cent and add this to the original rather than use the multiplier 1.2. While effective with a single change, this strategy soon becomes unwieldy when working with repeated percentage changes.

A common misconception is that a percentage change of 20 per cent followed by a percentage change of 20 per cent results in a percentage change of 40 per cent. While this can be demonstrated to be false, students may have a natural predisposition to think additively, and this intuition needs to be considered when working with repeated percentage changes.

Students need to	Guidance, discussion points and prompts
Understand that, in repeated percentage changes, the 'whole' changes at each stage *Example 1:* *Look at the diagram below.*	*Example 1* gives an opportunity to discuss and unpack some of the **language** and misconceptions around repeated percentage change in a context that is relatively straightforward. Draw attention to the changing 'whole' each time by asking students to be clear about which area is half of which whole, particularly in part c. The **representation** here is a visual to support students' understanding of repeated percentage change, which is a concept that students often meet in contexts (such as money) but without a representation to support.

 a) *What percentage of the area of rectangle A is the area of rectangle B?* b) *Complete these statements about relative areas with the appropriate percentage:* *B is ___ of A* *C is ___ of B* *D is ___ of C* c) *Complete these statements about relative areas with the appropriate percentage:* *C is ___ of A* *D is ___ of B* *D is ___ of A* d) *Mo says, 'The area of rectangle D is 50% of 50% of 50% of 50% of the whole rectangle.' Do you agree?*	You might find that demonstrating this visual 'live', by drawing the rectangle on the board and creating parts A to D in order, as needed, further supports understanding. Cutting a cake or piece of toast repeatedly in half is another context that might be useful to link to here.
Example 2: *A shop has a closing-down sale. Each day it reduces items by 50%.* *Ken says, 'After two days they'll have reduced everything by 100% and will give it away for free!'* a) *Do you agree with Ken?* b) *What will be the actual percentage reduction after two days? After three days?*	Part a of *Example 2* will expose those students that choose to work additively, while part b offers a direct connection to the **representation** used in *Example 1*. It is intended that these two examples be used together, with *Example 2* offering a different way of wording the same mathematics that was worked on in *Example 1* part d. *Examples 1* and *2* use halving to look at the fraction and percentage changes between situations. It might be argued that this is too familiar, and students are able to work intuitively without explicitly calculating the changes. What would change if the rectangles were each one-third or one-tenth of the other? Would this make students more likely to notice the underpinning structure?

Example 3: *Jon writes a number chain. Each number is 20 per cent greater than the one before it.*  *Jon's chain has a starting number of 5.* a) *How many increases will there be for the starting number to have doubled?* b) *Magne decides to try using 21 as a starting number.*  *Do you think Magne's chain will double faster or slower than Jon's? Or will the change of starting number make no difference?* c) *If the starting number is M, what will be in each of the first four 'clouds'?*  *How do you know when M has doubled?*	*Example 3* uses **variation** to focus on the multiplicative rather than the additive change. Students may intuitively think that a larger starting number will alter the number of repetitions needed to double that number. Attention can be drawn to the role of the multiplier, calculating, and writing the multiplier for each stage as a single value, as in part c, will draw particular attention to this.
Know that repeated percentage change can be represented by a single multiplier using index notation *Example 4:* *Percy and Monty are growing some different flowers and estimate that the height of each of their flowers will increase by a certain percentage every week of the summer.* *Percy wants to predict how tall a flower might be. He writes the calculation:* $7 \times 1.2 \times 1.2 \times 1.2 \times 1.2 \times 1.2 \times 1.2 \times 1.2$. a) *How many centimetres tall do you think the flower is at the start of the summer?* b) *For how many weeks does Percy estimate the plant will grow?* c) *What percentage does he estimate the flower will grow by each week?* *For a different flower, Monty writes the calculation:* 5×1.2^9.	*Example 4* uses a context of growth to unpack the role played by the different values in the expression. Students should appreciate that an expression is a **representation** of a mathematical relationship, and the role of the different variables can be explored and discussed. The shift to using index notation in part d, rather than writing the repeated multiplication, gives an opportunity to discuss and explore students' understanding of this notation (which is probably familiar) and to recognise its meaning in this context.

d) *Is Monty's flower originally taller or shorter than Percy's?* e) *Does Monty's flower grow faster than Percy's?* f) *Does Monty's flower grow for longer than Percy's?* g) *How do you know?*	
Example 5: *Complete as many cells as possible in the table below.*	*Example 5* uses a completion table to practise and **deepen** understanding of the role played by each term in the expressions used when calculating with repeated percentage change. If students do not have a solid grasp of the single multiplier, then interpreting the percentage change when decreasing may prove particularly challenging, since the value of this term is not explicit in the formula; rather it needs to be calculated by subtracting the multiplier from one.

In words	Expression	Original quantity	Percentage change	Number of repetitions
Increase 27 by 15%, 3 times.	27×1.15^3	27	15% increase	3
Increase 30 by _%, _ times.	30×1.81^8			
		12	16% increase	12
	17×0.75^4		25% decrease	
Decrease 106 by 10%, 6 times.				
	77×0.77^{47}			

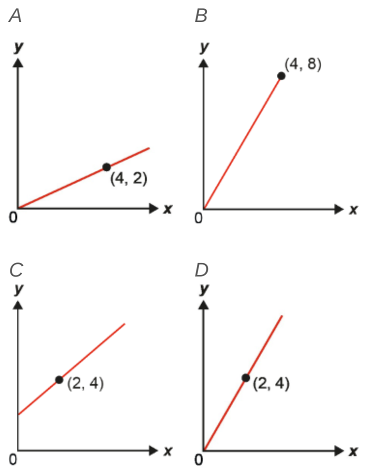
8.2.2.3 Know how the key features of a proportional relationship are represented graphically

Common difficulties and misconceptions

When a linear relationship is expressed algebraically in the form $y = mx + c$, students can identify 'm' and 'c' as being the 'gradient' and 'y-intercept' of a straight-line graph, but they may not necessarily recognise that when 'c' is zero, proportionality is represented. A straight-line graph going through (0, 0) is a key feature of the graphical representation of a proportional relationship between x and y, ($y \propto x$ or $y = kx$). The use of 'k' to describe the constant of proportionality in the algebraic representation ($y = kx$) and 'm' to describe the gradient of a straight-line graph ($y = mx + c$) can hinder their ability to connect the rate of change with the functional multiplier in a proportional relationship.

Students need to recognise that linear graphs that cross the x- or y-axis other than at the origin are not representing proportionality. Giving time to interpreting the graphical features of non-proportional relationships, including special cases, supports them in developing their understanding of functional relationships. Students often struggle to identify that a horizontal line has a gradient of zero, for example, and that the gradient of a vertical line is undefined. Once established, students may associate graphical representations of proportional relationships as straight-line graphs with positive gradients in the first quadrant, where both the x- and y-coordinates are positive. Presenting students with graphs of a variety of proportional and non-proportional relationships beyond this provides an opportunity to deepen their understanding of what does and doesn't constitute a proportional relationship.

Students often assume that a proportional relationship will always produce a linear graph of y against x that passes through the origin and so struggle to recognise why relationships between y and x that are represented by curves can also be proportional. Students should familiarise themselves with the graphical representations of proportional relationships such as $y = kx^2$, $y = kx^3$, $y = \sqrt{x}$, as well as graphs of inversely-proportional relationships, so that they can recognise their shapes. However, to fully understand why these relationships are proportional, in a way that does not rely on memorising the shapes of the curves when represented graphically, requires an appreciation of the key features of a proportional relationship. It is useful to spend time thinking about non-linear graphs of proportional relationships and considering how they can be clearly identified as a proportional relationship from their algebraic equation. This gives students the opportunity to think more deeply about the underlying multiplicative structure and how this can be represented graphically for all proportional relationships.

Students need to	Guidance, discussion points and prompts
Understand that the gradient of a straight-line graph that goes through the origin represents the constant of proportionality *Example 1:* *Which of the graphs A to D represent the proportional relationship $y = 2x$? Explain how you know.* 	In *Example 1*, students explore the link between the constant of proportionality and the gradient of a straight line. It is important that they are familiar with and use this **language** when explaining what the 2 describes in this example. They may be accustomed to referring to the gradient as m, as they have previously described straight-line graphs algebraically in the form $y = mx + c$. The four graphs expose some common misconceptions. Students who select graph A may be thinking of gradient as the change in x divided by the change in y, rather than the change in y divided by the change in x. Students who select graph D but not graph B may be viewing the relationship between x and y as additive rather than multiplicative. Students who select graph C may not fully understand the difference between a non-proportional relationship that is linear and a directly-proportional relationship. The relationship between y and x when x is 2 and y is 4 is significant, as there is an additive relationship of '+ 2' as well as a multiplicative relationship of '× 2'. Further questioning may be needed to establish students' thinking about the relationship between the two variables. What might be revealed about students' understanding by asking them to identify another point on the graph? How would asking students to identify the equations of the lines give insights into their understanding of the relationship between the variables y and x?

<table>
<tr>
<td>
Use graphical representations to distinguish between relationships that are directly-proportional and non-proportional linear relationships
Example 2:
Determine which of the graphs below represent a proportional relationship and write an equation for the relationship between x and y.
</td>
<td>
In Example 2, the identification of proportional relationships is further developed to explore the way in which the relationships between variables x and y are expressed algebraically. Students' algebraic descriptions can provide an insight into their understanding of multiplicative relationships.
The use of 2 and 4 for the values of x and y is intended to provoke a discussion about how the value of y can be obtained from the value of x. Students who are thinking additively may describe graph F as $y = x + 2$, for example, rather than $y = 2x$. Three of the graphs represent proportional relationships and the inclusion of graph H aims to expose the common misunderstanding that a proportional relationship always has a positive constant of proportionality.
Double number lines are commonly used as a representation for proportional relationships of the form $y = kx$, where the zeros on the two lines are aligned and the scales are both linear. Thinking about the double number line representation for the relationships represented by the graphs may help to identify the difference between proportional and non-proportional relationships and reinforce the significance of a proportional graph going through the origin/aligned zeros on the two number lines. Less obvious but significant features include:

The multiplicative relationship between the lines is not constant for a non-proportional relationship (for graph G, $6 \div 4 = 1.5$, whereas $0 \div 1 \div = 0$), so 'between the lines' strategies do not work.
For a non-proportional relationship, the multiplicative relationships between the numbers on the top number line are not consistent with those on the bottom number line. For graph G, for example, 4 is quadruple 1, but 6 is not quadruple 0, so 'along the lines' strategies do not work.

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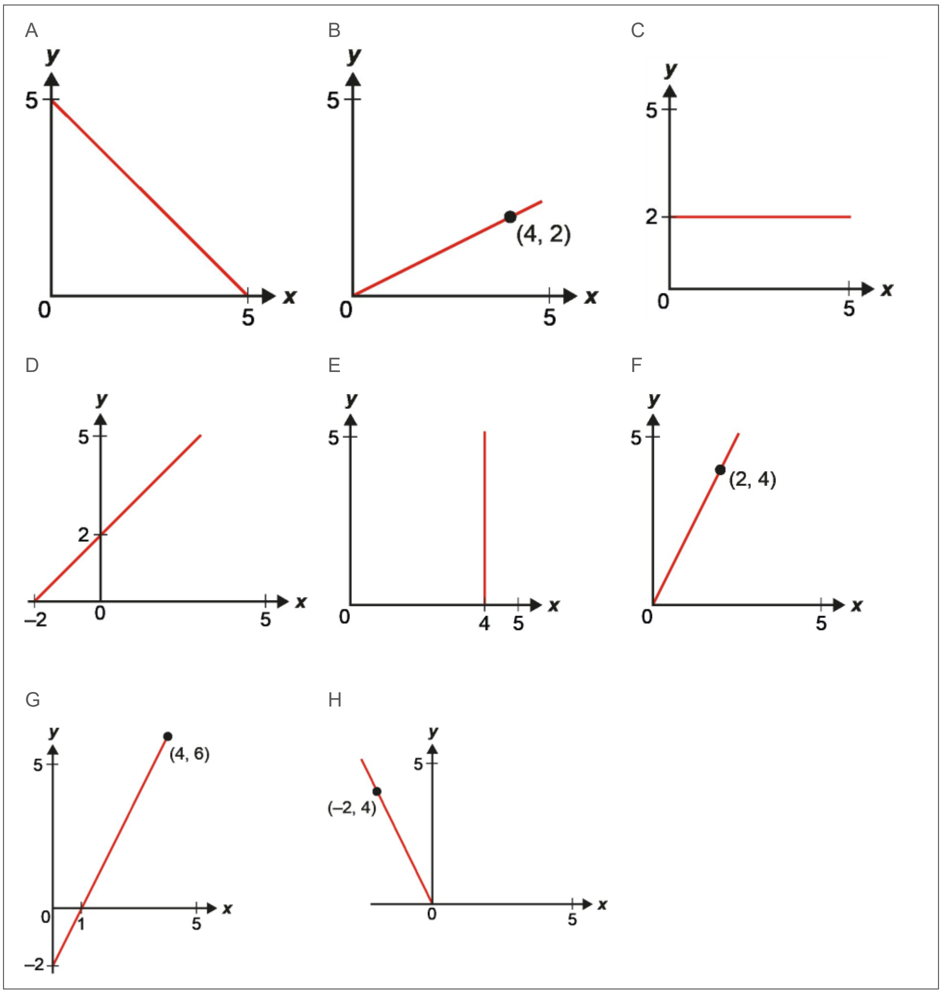
A
y
5
0
5
x
B
y
5
(4, 2)
0
5
x
C
y
5
2
0
5
x
D
y
5
2
–2
0
5
x
E
y
5
0
4
5
x
F
y
5
(2, 4)
0
5
x
G
y
5
(4, 6)
0
1
5
x
–2
H
y
5
(–2, 4)
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x

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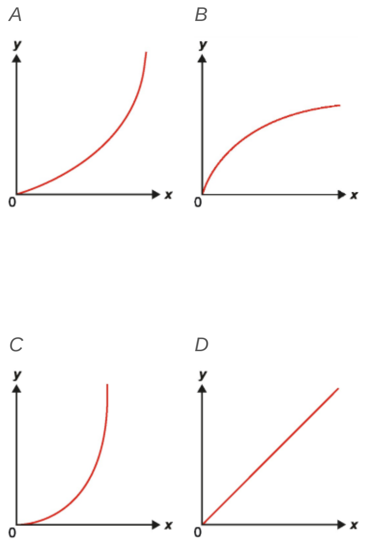
<td>**Be familiar with the graphical representations of standard proportional relationships**

Example 3:

These graphs show four different proportional relationships.



Match each graph with the correct description of direct proportionality below.

$y \propto x$,

$y \propto x^2$,

$y \propto x^3$

$y \propto \sqrt{x}$.</td>
<td>*Example 3* explores characteristics of graphical **representations** for relationships that are directly proportional. At Key Stage 4, students are expected to be familiar with graphs of quadratic functions, simple cubic functions and graphs such as $y = \sqrt{x}$.

Recognising graphs of these types of functions* that represent a proportional relationship is an important stage in the development of students' understanding.

What types of questions might you use to draw students' attention to the distinct characteristics of the different graphs? Consider asking:

'What's the same and what's different?'

'Which two graphs are most alike and how is it possible to distinguish between them?'

'How could you describe the relationship between graphs B and C?'

'What do the graphs look like in the other three quadrants?'

Once the graphs have been matched to the corresponding algebraic description, **deepening** students' understanding could be achieved by considering graphs A, B and C and the ways in which the graphical representations make the directly proportional relationships visible.

Consideration of the relationship between y and x^2, y and x^3 and y and $\sqrt{x}$, and how this relates to the relationship represented by graph D, enables students to think more deeply about the common structure of all the proportional relationships.

**Although the square root of x has a positive and a negative value, we consider $\sqrt{x}$ to be just the positive (principal) value. If both the positive and the negative values are considered, then the result is not a function, as each value in the domain maps to two values in the range (except 0).*</td>
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<td>*Example 4:*

The three graphs show different proportional relationships.

Match the graphs with the correct descriptions of inverse proportionality.

$y \propto \frac{1}{x^2}$ $y \propto \frac{1}{x^3}$ $y \propto \frac{1}{\sqrt{x}}$</td>
<td>Inverse proportionality can be represented graphically by reciprocal curves. *Example 4* enables students to think about the inversely proportional relationships that produce standard graphical relationships.

To be able to distinguish the graphs, it is necessary to consider all four quadrants, rather than just the positive quadrant as in *Example 3;* the **variation** here is intended to give an opportunity to discuss this.

Asking students what is the same and what is different about the graphs can help to establish the key features of graphical representations of inversely-proportional relationships.</td>
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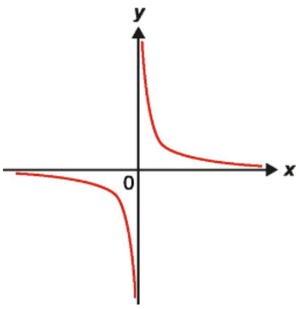
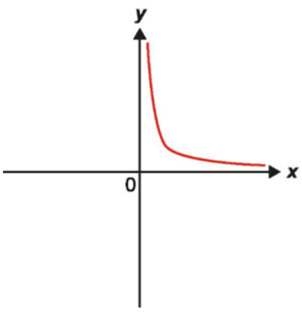


F



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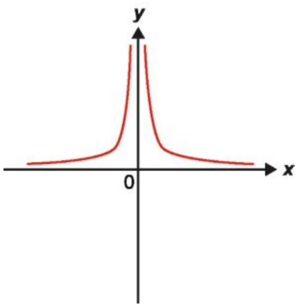

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You may like to ask: '*How do the graphs demonstrate the key characteristic of inverse proportionality that when one variable doubles (or is multiplied by any constant) the other variable halves (is divided by the same constant)?'*

Ensure that students recognise that the value of *y* is undefined when $x = 0$ and understand why this is what it means.

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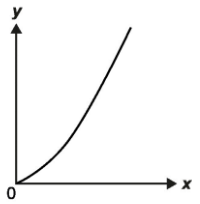
Appreciate the underlying multiplicative structure of proportional relationships $A = kB$ where $B = x^n, x > 1$

Example 5:

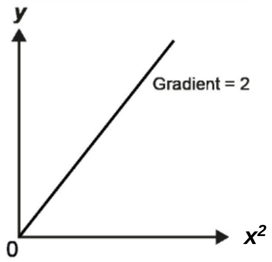
Shaun draws a sketch of the graph

$y = 2x^2$.



He says it does not represent a proportional relationship because it is a curve, not a straight line.

Sarah says it is a proportional relationship with constant of proportionality 2 and sketches the graph below.



Who is correct, Shaun or Sarah?

Example 5 uses graphical **representations** to expose the underlying multiplicative structure of proportional relationships.

The graph of $y = 2x^2$ has been used, as the shape of the curve is likely to be familiar to students. They are less likely to begin by questioning whether Shaun has sketched the graph correctly if some familiarity with the curve exists.

In this example, a misconception is exposed and the question wording engages with the misconception. Students are often more confident discussing misconceptions that are perceived as someone else's, even though they may share the same incorrect thinking.

It is likely that students who agree with Shaun view non-linear proportional relationships as being distinct from linear relationships, as the graphical representations have different characteristics.

Sarah's thinking highlights that when we consider non-linear proportional relationships, it is not necessarily the relationship between x and y that we are interested in. If, instead of graphing y against x, we graph, as in this example, y against x^2, the key features of a proportional relationship (a straight-line graph passing through the origin) are displayed.

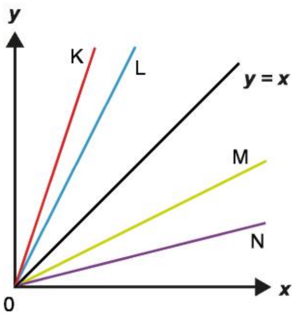
Discuss the validity of Sarah's sketch as a graphical representation. What are the issues with having an axis labelled as x^2 rather than x?

Students could be challenged to consider other non-linear proportional relationships as an opportunity for **deepening** their understanding. Can Sarah's approach be applied to those as well?

8.2.2.4 Interpret the gradient as a rate of change

Common difficulties and misconceptions

Students often associate gradient with a way of describing the slope of a linear graph and recognise that as the magnitude of the gradient increases, the line gets steeper. The relationship between gradient and rate of change may not have been established, and students will benefit from the opportunity to calculate gradients of graphs within a variety of contexts to enable them to interpret the gradient as a rate of change of one variable in relation to another. A commonly-held misconception is that gradients are always positive. Providing students with opportunities to explore relationships that have a negative rate of change and be able to interpret this within a real-world context can help to expose and address this incorrect assumption.

Students need to	Guidance, discussion points and prompts
Understand the relationship between the constant of proportionality and the gradient for directly-proportional linear relationships *Example 1:* *Match the linear functions with the corresponding graphical representation.* *A:* $y = 2x$ *B:* $y = 3x$ *C:* $y = \frac{1}{2}x$ *D:* $y = \frac{x}{4}$  	The purpose of *Example 1* is to establish students' existing knowledge of gradient as a measure of slope and identify the relationship between the constant of proportionality and the gradient of the line. Students may already be familiar with dividing the change in the y-axis by the change in the x-axis to find the gradient, and it is this conceptual understanding that can be developed to recognise the effects an increase/decrease in the rate of change has on the graphical representation. Explore and **deepen** students' understanding by asking questions such as: *'How is the gradient of the graph identifiable from the algebraic representations?'* *'How is identifying the constant of proportionality for* $y = \frac{x}{4}$ *different to identifying it for the other three linear functions*?' *'Which straight-line graph shows the greatest rate of change?'*

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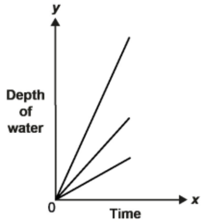
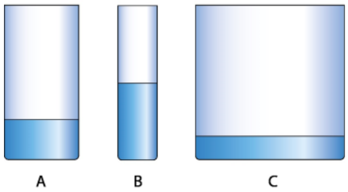
Appreciate the rate of change as a description of how one quantity changes in relation to another

Example 2:

Three containers are being filled with water at a constant rate. The graph shows the change in depth of water over time.





Label each line of the graph with the letter of the corresponding container.

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Example 2 helps students to think about how two quantities vary together where the rate of change is not explicitly given and cannot be calculated. Providing a context for exploring rate of change without considering specific values supports students in **deepening** their understanding of how the graphical representation relates to the rate at which one quantity is changing in relation to another.

To distinguish the graphs, students have to think about the rate at which the depth of the water is changing over time. Container B is half as wide as container A and container A is half as wide as container C. Students relate the width of the container to how quickly or slowly the depth of water changes as water is added at a constant rate. Encourage students to explain their reasoning:

'Why is the depth of water in the three containers at different levels when they have all been filling for the same amount of time?'

'Which container will take the longest to fill? How is this represented on the graph?'

'Describe what the graph will look like when all three containers are full.'

'If the three containers held the same volume of water, what could we say about the time it would take for them to fill?'

'Which container has the greatest rate of change of depth over time?'

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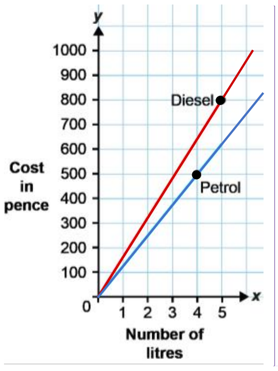
Understand that the gradient is equal to the unit rate

Example 3:

The graph shows the price of petrol and diesel:



Calculate the difference in price per litre for petrol and diesel.

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Example 3 provides students with an opportunity to compare two gradients and is designed to highlight how the rate of change gives the unit rate for directly proportional linear relationships.

The points (4, 500) and (5, 800) have been marked on to provide exact values that can be used to calculate the gradients of the two lines. It is likely that students will divide 500 by 4 and 800 by 5 to find the two gradients and then complete a subtraction to find the difference in price per litre for petrol and diesel.

To establish that the rate of change given by finding the gradient is the same as the cost in pence of one litre of petrol/diesel (unit rate), you could ask: *'How else could the graph be used to find the unit rate for petrol and diesel?'*

The graph has been designed in such a way that it is not possible to read off accurately the cost in pence of one litre of petrol and one litre of diesel and so the **representation** is intended to challenge students to calculate rather than estimate.

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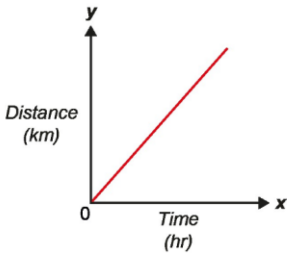
Commented [RP1]: AW - need to be accurate BUT it is intended that students can't just read off the graph and have to calculate so keep the grid this size rather than anything more fine

Determine what the rate of change represents for a variety of contexts

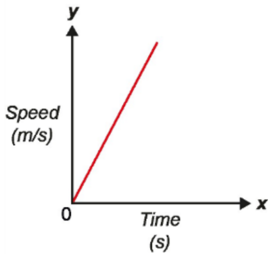
Example 4:

Describe what the gradient represents for each of graph:

A



B



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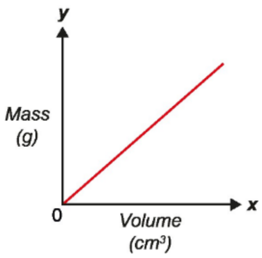


At Key Stage 4, students develop their ability to change freely between compound units in numerical and algebraic contexts. *Example 4* is an opportunity to explore the graphical **representations** of these relationships. The representation focuses on the units rather than numerical values, although students may find it helpful to think about specific values to help them to understand what the graph shows.

The use of a context enables students to give meaning to the rate of change and provides a way of describing it. For example, the change in distance over time gives the speed in kilometres per hour. The **language** of 'per standard unit' is important and compound measures provide examples of this relationship between measures of different types.

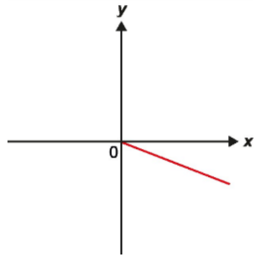
The rate of change is constant for all four graphs and so a directly-proportional relationship has been represented. Think about **deepening** students' thinking by asking whether any other directly-proportional relationships can be represented for these compound measures? How can compound measures be used to demonstrate inverse proportional relationships?

Identify negative rates of change

Example 5:

Which of these graphs could be considered the 'odd one out'?

A



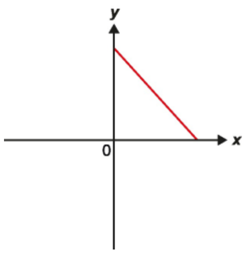
Example 5 provides an opportunity for students to recognise and identify a negative rate of change.

There is more than one possible graph that could be considered as being the 'odd one out'. For example:

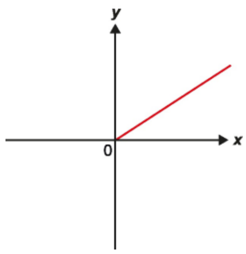
- Graph C is the only graph that has a positive rate of change. As y increases, so does x. For graphs A, B and D, as x increases, y decreases.
- Graph B represents a non-proportional linear relationship and could be considered the 'odd one out', as the other three graphs represent relationships that are directly proportional.

What are the benefits of contrasting the linear relationship shown in graph B with the directly-proportional relationships represented by graphs A and D? How does what the rate of change represents differ?

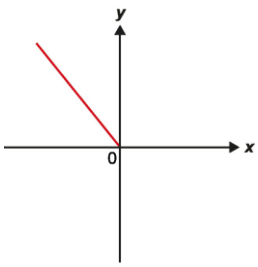
B



C



D

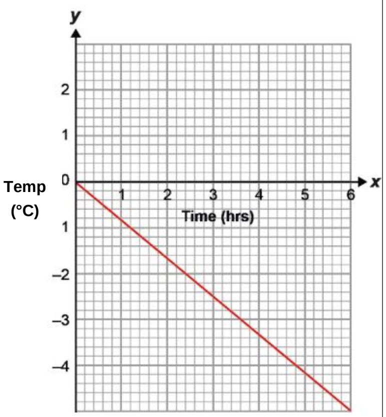


Interpret negative rates of change

Example 6:

The outside temperature in °C is observed and recorded hourly overnight and plotted on a graph.

Describe the rate at which the temperature is falling.



Example 6 explores a negative rate of change for a proportional relationship using **language** as a structure to probe understanding. It has been worded in such a way that the rate of change should be given as a positive value, as it describes the rate at which the temperature is <u>falling</u>.

It is common when describing negative rates of change to refer to them as a positive value, e.g. a decrease of 5°C per hour rather than an increase of -5°C per hour. Discuss why this might be with your colleagues. Would there ever be a context where we would want to describe the rate of change as a negative value?

There are several contexts that can be used to demonstrate a negative rate of change for a non-proportional linear relationship (for example, constant deceleration, walking back to a starting point at a constant speed). Finding contexts that can be used to represent a negative rate of change for a proportional relationship may be more difficult. What other contexts might be used to demonstrate a negative rate of change?

8.2.3.3 Interpret the gradient of the tangent to a curve as the rate of change of the relationship represented by the curve at that instant

Common difficulties and misconceptions

Making the connection between the gradient of a straight line and the gradient of the tangent to a curve requires students to understand that they are describing the rate of change of the relationship represented by the curve at a particular point. This provides them with an opportunity to recognise that finding the gradient of a linear graph can be applied to finding the constantly changing gradient of a curve. The idea of drawing a tangent to a curve is introduced for the first time in Key Stage 4 and students often assume that a tangent will not cross the curve. Recognising that, when extended, a tangent may cross the curve, while establishing that a tangent is a line that touches a curve at a point, is important in helping students to understand the features of a tangent, and how a tangent to a curve matches the slope of the curve at that particular point. Students often assume that a tangent to a curve must always pass through the origin; it is important that they experience graphs where the shape of the graph does not facilitate this.

<table>
<tr><th>Students need to</th><th>Guidance, discussion points and prompts</th></tr>
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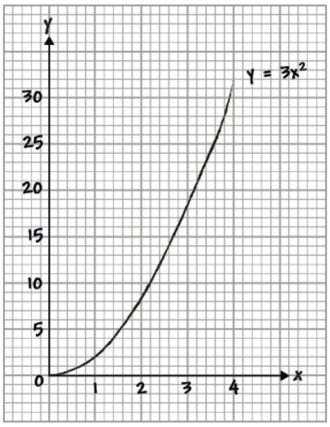
Appreciate that the gradient of a curve constantly changes

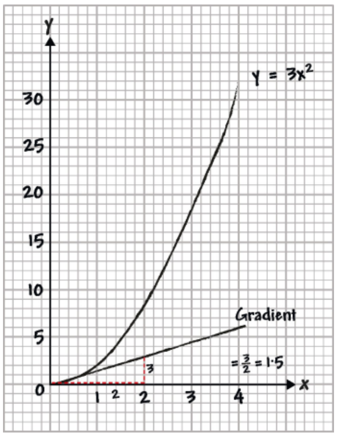
Example 1:

Edward and Esther are thinking about the gradient of the function $y = 3x^2$.

Edward draws a graph and says that the gradient of the curve is 3 as 3 is the constant of proportionality.



Esther draws a tangent to the curve and concludes that the gradient is 1.5.



Explain why both Edward and Esther are wrong.

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Example 1 aims to expose two common misconceptions, with the prompts intended for **deepening** students' thinking around gradients and curves. When exploring linear relationships that are directly proportional, the association between the constant of proportionality and the gradient of the line can become well established. Students may assume that this extends to all proportional relationships, resulting in them identifying the constant of proportionality as the gradient of a curve.

Discuss other factors that may influence students' association of constant of proportionality with the gradient of non-linear proportional relationships. When comparing the graphical representations of $y = 3x^2$, with $y = 2x^2$, for example, students often interpret the stretch parallel to y-axis scale factor 1.5 with an increase in the steepness of the curve. How might this cause a misconception to be established?

Although for most points on a curve it is not possible, students often assume that the tangent to a curve must pass through the origin. The positioning of the tangent on Esther's graph prompts a discussion about why she may have chosen to draw a tangent at the point when $x = 0.5$, rather than elsewhere on the curve.

When presented with a non-proportional, non-linear relationship (for example, $y = 2x^2 + 3$) how likely are students to identify the gradient of the curve as being 2? How does this compare to non-proportional linear relationships? What is the underlying misunderstanding that Edward and Esther both have about the gradient of a curve?

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Understand that the gradient of the tangent to a curve provides an estimate for the rate of change

Example 2:

Solomon and Stacey are exploring the gradient of the graphical representation of the function $y = \frac{3}{x}$. They each use the graph to find the rate of change when $x = 2$.

Their workings are shown below.

Why do Solomon and Stacey have different answers?

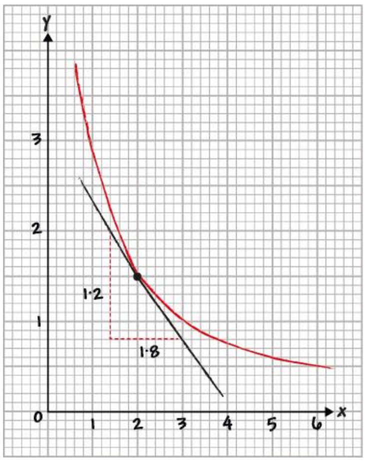
Example 2 establishes that the method of using the gradient of a tangent to the curve at a certain point to find the rate of change provides an estimate rather than an exact solution.

There are two main factors that affect the value identified as the rate of change:

- The positioning of the tangent. While a tangent to a curve is a unique straight line that meets the curve at exactly one point, when drawn by hand some variation is likely to occur.
- The variation of values read off the graph and used to determine the change in y over the change in x. When reading values off a graph, even if a consistent tangent is being used (as in Example 2), there are likely to be some differences in the values identified, resulting in slight variation in the value of the rate of change obtained.

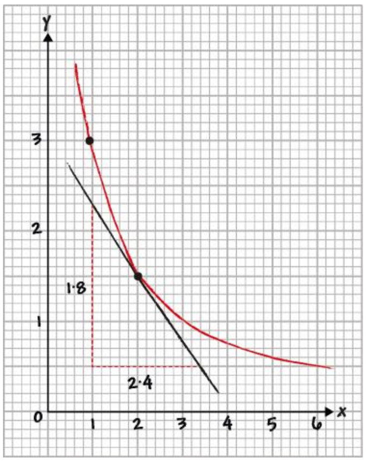
An alternative approach to understanding the variation in values obtained for the rate of change may be to provide students with identical graphs and ask them to find the rate of change of the curve at a particular point. When the values obtained are not unique, discuss why this may be the case.

Solomon's calculation



Rate of change = $\frac{-1.2}{1.8} = \frac{-2}{3}$

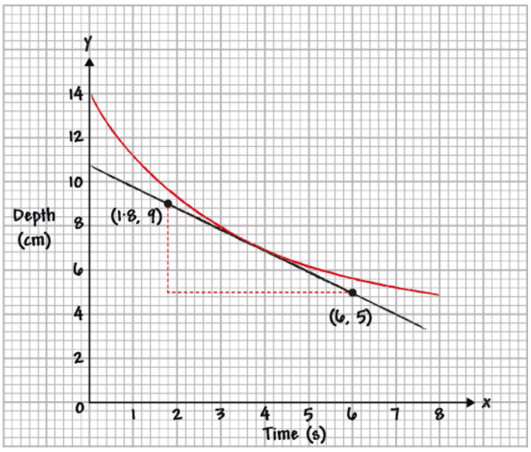
Stacey's calculation



Rate of change = $\frac{-1.8}{2.4} = \frac{-3}{4}$

Know that a tangent can be drawn at any point on the graph and gives the rate of change of the curve at that point *Example 3:* *A sink is being emptied of water. The graph shows the depth of water over time.* *Shobhna draws a tangent and finds values for the x_1, x_2, y_1 and y_2.* a) *What does Shobhna need to do to find the rate at which the depth of water is changing?* b) *For what time has she found this rate of change?* c) *Suggest three other times she might choose to draw tangents at. How would the gradients, and hence the rate of change, differ at these times?*	Use *Example 3* to provide insight into students' understanding of a negative rate of change. Students need to determine how to report the rate of change. Do they describe the depth of water as changing at a rate of −0.95 cm per second or decreasing at a rate of 0.95 cm per second? Challenging students to think of other contexts where the rate of change is negative and the different ways that the rate of change can be described may **deepen** their understanding. For example, a negative acceleration can be described as a deceleration. What other contexts can student think of? Students often assume that tangents can only be drawn at integer values and may solely identify such times. It is important to establish that a tangent can be drawn at any point on the curve and gives the rate of change of the curve at that point. This builds a foundation for developing an understanding of the gradient function which some students will go on to explore at Key Stage 5.

Shobhna's graph



Using these materials

Collaborative planning

Although they may provoke thought if read and worked on individually, the materials are best worked on with others as part of a **collaborative professional development** activity based around planning lessons and sequences of lessons.

If being used in this way, is important to stress that they are not intended as a lesson-by-lesson scheme of work. In particular, there is no suggestion that each key idea represents a lesson. Rather, the fine-grained distinctions offered in the key ideas are intended to help you think about the learning journey, irrespective of the number of lessons taught. Not all key ideas are of equal weight. The amount of classroom time required for them to be mastered will vary. Each step is a noteworthy contribution to the statement of knowledge, skills and understanding with which it is associated.

Some of the key ideas have been extensively exemplified in the guidance documents. These exemplifications are provided so that you can use them directly in your own teaching but also so that you can critique, modify and add to them as part of any collaborative planning that you do as a department. The exemplification is intended to be a starting point to catalyse further thought rather than a finished ‘product’.

A number of different scenarios are possible when using the materials. You could:

- Consider a collection of key ideas within a core concept and how the teaching of these translates into lessons. Discuss what range of examples you will want to include within each lesson to ensure that enough attention is paid to each step, but also that the connections between them and the overall concepts binding them are not lost.
- Choose a topic you are going to teach and discuss with colleagues the suggested examples and guidance. Then plan a lesson or sequence of lessons together.
- Look at a section of your scheme of work that you wish to develop and use the materials to help you to re-draft it.
- Try some of the examples together in a departmental meeting. Discuss the guidance and use the PD prompts where they are given to support your own professional development.
- Take a key idea that is not exemplified and plan your own examples and guidance using the template available at Resources for teachers using the mastery materials | NCETM.

Remember, the intention of these PD materials is to provoke thought and raise questions rather than to offer a set of instructions.

Solutions

Solutions for all the examples from *Theme 8 Proportional reasoning* can be found at

https://www.ncetm.org.uk/media/4gdbakqz/ncetm_ks4_cc_8_solutions.pdf