MathsHUBS Resources

National Centre for Excellence in the Teaching of Mathematics

Mastery Professional Development

Multiplication and Division

2.27 Scale factors, ratio and proportional reasoning

Teacher guide | Year 6

Teaching point 1:

Multiplication and division can be used to calculate unknown values in correspondence (cardinal comparison) problems.

Teaching point 2:

Multiplication and understanding of correspondence can be used to calculate the number of possible combinations of items.

Teaching point 3:

Scaling can be used to make and interpret maps.

Teaching point 4:

There is a proportional relationship between the dimensions of similar shapes; if the scale factor and the dimensions of one of the shapes is known, the dimensions of the similar shape can be calculated; if the dimensions of both of the shapes are known, the scale factor can be calculated.

Overview of learning

In this segment children will:

- use bar modelling and ratio grids to reason about multiplicative relationships between two or more cardinal quantities, solving problems such as *'For every five blue marbles, there are three red marbles. If there are fifteen blue marbles, how many red marbles are there?'*
- explore correspondence problems in the context of calculating the number of possible combinations of certain items, such as *'If Megan has four coats and three hats, how many different outfits can she make?'*
- extend their understanding of scaling measures to:
  - making and interpreting maps
  - scaling the dimensions of shapes
  - calculating the scale factor relating two similar shapes.
    (Two shapes are 'similar' if one can be transformed into the other by scaling; there may also be a reflection, translation or rotation.)

This segment builds on children's understanding of:

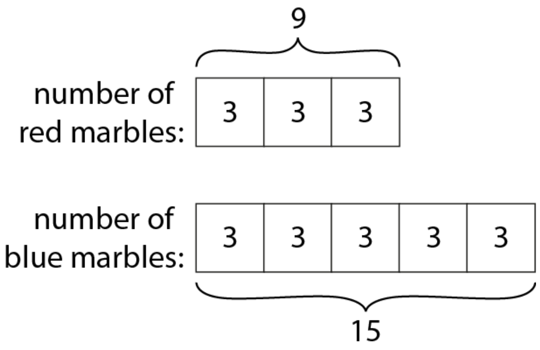
- scaling cardinal quantities (segment *2.13 Calculation: multiplying and dividing by 10 or 100*); for example, *'Emily has two pencils; Jamie has ten times as many. How many pencils does Jamie have?'*
- scaling measures, and relating scaling by a unit-fraction scale factor to division by the denominator of the scale factor (segment *2.17 Structures: using measures and comparison to understand scaling*); for example:
  - *'The plain ribbon is three times the length of the spotty ribbon.'*
  - *'The spotty ribbon is one-third times the length of the plain ribbon.'*

In *Teaching point 1*, children explore contexts where a multiplicative relationship (ratio) between two or more cardinal quantities is given and then used to solve problems. Initially, bar modelling is used to provide a visual representation of the multiplicative relationships, then children move to the more abstract *'ratio-grid'* method; for example:

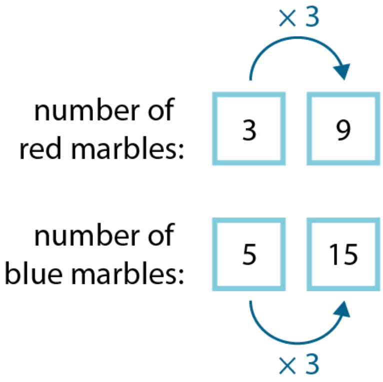
'Bijan has some marbles. For every five blue marbles, he has three red marbles.'

- *'If Bijan has fifteen blue marbles, how many red marbles does he have?'*
- *'How many marbles does he have altogether?'*

Bar modelling:



Ratio grid:



In *Teaching point 2*, children learn how to calculate the number of possible combinations of two types of item, for example, *'If Megan has four coats and three hats, how many different outfits can she make?'* Children begin with a simple problem (for example, one coat and one hat) and gradually increase the number of items (one coat and two hats; one coat and three hats; two coats and three hats…), until they come to the understanding that the number of combinations can be calculated by multiplying together the number of each type of item (*number of coats* × *number of hats* = *number of combinations*); the process of 'working up' from one of each item is critical to developing a deep understanding of the mathematics involved.

In *Teaching points 3* and *4*, children return to scaling lengths (first explored in segment *2.17*). *Teaching point 3* introduces the term *'scale factor'* to describe the multiplicative relationship between distances on a map (or scale drawing) and corresponding distances in the 'real world', and children convert from one to the other. In a similar progression to that used in *Teaching point 1* (for cardinal quantities), initially a visual representation of the relationships is used in the form of double number lines, and then ratio grids are introduced as a more efficient, abstract way of working.

In *Teaching point 4*, children continue to use the term 'scale factor', and are introduced to the term *'ratio'*, to describe the multiplicative relationship between the dimensions of similar shapes. Children learn the defining features of similar shapes, and they also learn to use scaling to transform between similar shapes, including both regular and irregular polygons.

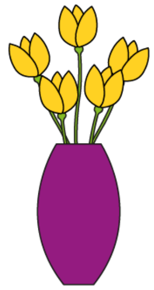
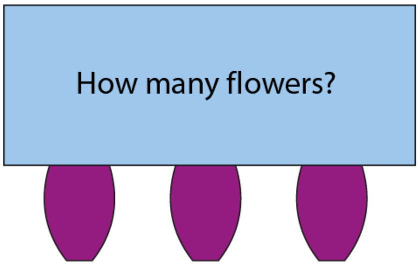
Note: for the representations in *Teaching points 3* and *4*, measurements have been drawn at actual size (when the unit is specified) unless stated otherwise.

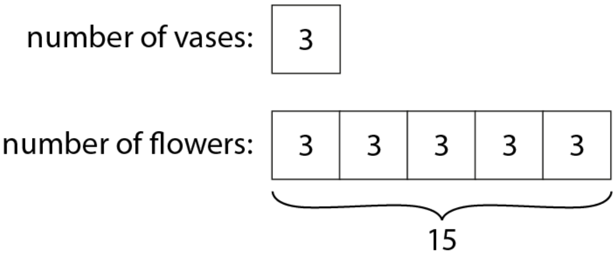
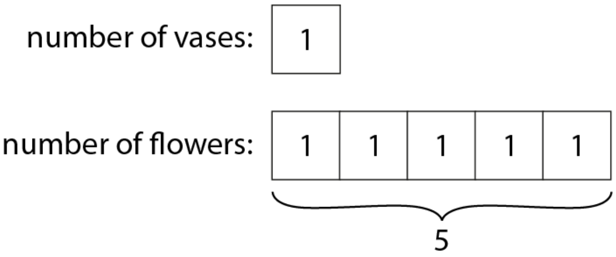
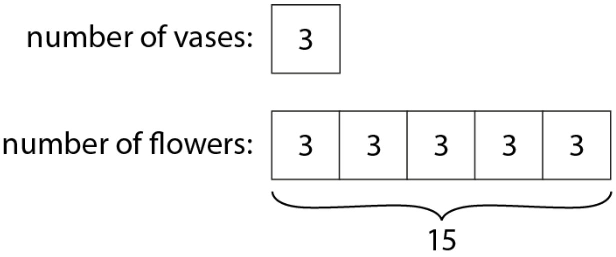
An explanation of the structure of these materials, with guidance on how teachers can use them, is contained in this NCETM podcast: www.ncetm.org.uk/primarympdpodcast. The main message in the podcast is that the materials are principally for professional development purposes. They demonstrate how understanding of concepts can be built through small coherent steps and the application of mathematical representations. Unlike a textbook scheme they are not designed to be directly lifted and used as teaching materials. The materials can support teachers to develop their subject and pedagogical knowledge and so help to improve mathematics teaching in combination with other high-quality resources, such as textbooks.

Teaching point 1:

Multiplication and division can be used to calculate unknown values in correspondence (cardinal comparison) problems.

Steps in learning

	Guidance	**Representations**
1:1	In segment *2.13 Calculation: multiplying and dividing by 10 or 100*, children learnt how to find 10 or 100 times a quantity; they learnt how to describe and use multiplication to solve correspondence problems such as *'Emily has two pencils; Jamie has ten times as many. How many pencils does Jamie have?'* Similarly, children learnt to describe and use division to solve correspondence problems where the larger quantity is known and the smaller quantity is unknown. At that point, children's understanding of such problems was based on the inverse of 'ten times as many'/'ten times the size'; for example, they considered the problem '*Jamie has twenty pencils; he has ten times as many pencils as Emily. How many pencils does Emily have?'* Note that fractional language (i.e. *'Emily has one-tenth as many pencils as Jamie.'*) was not used at that stage, although it was introduced in the context of scaling measures in segment *2.17 Structures: using measures and comparison to understand scaling.* In this teaching point, children extend their understanding of correspondence problems to quantites that are in ratios other than 1:10 and 1:100. They also extend their understanding from segments *2.13* and *2.17*, to link division with fractional language when describing/finding smaller quantities in terms of larger quantities. Begin by looking at a correspondence problem that children are already familiar with, but now using the	*'For every one vase, there are five flowers.'*  number of vases: ☐ number of flowers: ☐☐☐☐☐ *'If there are three vases, how many flowers are there?'*  - *'For every one vase, there are five flowers.'* $1 \times \mathbf{5} = 5$ number of vases: 1 number of flowers: $\underbrace{1 \quad 1 \quad 1 \quad 1 \quad 1}_{5}$

	language ***'For every ____ ____, there are ____ ____.'*** For example: *'For every one vase, there are five flowers. If there are three vases, how many flowers are there?'* Use the stem sentence above to describe the situation, write the corresponding multiplication equations ($1 \times 5 = 5$ and $3 \times 5 = 15$) and represent the relationships using bars, as shown opposite. Write the multiplier (here, '5') as the second factor, to draw attention the structure, and use the language of '*multiplied by*' and '*times the size*' to describe the relationship between the numbers in the equations.	• *'So for three vases, there are fifteen flowers.'* $3 \times \mathbf{5} = 15$  • *'Three multiplied by five is equal to fifteen.'* • *'Fifteen is five times the size of three.'*
1:2	Next, take the same example, but look at the fact that, if we know how the number of flowers corresponds to the number of vases, as well as the total number of flowers, we can calculate the number of vases using division. This time, write the corresponding division equations and multiplication by a fraction, and link to the original multiplication equations from step *1:1*.	*'For every one vase, there are five flowers. If there are fifteen flowers, how many vases are there?'* • *'For every one vase, there are five flowers.'* • *'For every five flowers, there is one vase.'* $1 \times \mathbf{5} = 5$ $5 \div \mathbf{5} = 1$ $5 \times \frac{\mathbf{1}}{\mathbf{5}} = 1$  • *'So, for three vases, there are fifteen flowers.'* • *'For fifteen flowers, there are three vases.'* $3 \times \mathbf{5} = 15$ $15 \div \mathbf{5} = 3$ $15 \times \frac{\mathbf{1}}{\mathbf{5}} = 3$  • *'Three multiplied by five is equal to fifteen.'* • *'Fifteen divided by five is equal to three.'* • *'Fifteen multiplied by one-fifth is equal to three.'* • *'Three is one-fifth times the size of fifteen.'*

1:3

Now introduce a different context using the ***'For every ___ ___ ,there are ___ ___'*** structure, but now with the larger quantity at the start of the sentence; for example: *'For every ten grapes that Ralph eats, Lily eats one.'*

Encourage children to look at the relationship between the numbers and to describe the context 'both ways', supported by a bar model, as shown below. Then pose questions based on this relationship:

- *'If Ralph eats twenty grapes, how many does Lily eat?'*
- *'If Lily eats three grapes, how many does Ralph eat?'*

Draw out the fact that Lily always eats one-tenth the number of grapes that Ralph eats, and that Ralph always eats ten times as many as Lily.

'For every ten grapes that Ralph eats, Lily eats one.'

- *'Ralph eats ten times as many grapes as Lily.'*
- *'Lily eats one-tenth as many grapes as Ralph.'*
- *'For every one grape that Lily eats, Ralph eats ten'*
- *'For every ten grapes that Ralph eats, Lily eats one.'*

- *'If Ralph eats twenty grapes, how many does Lily eat?'*
- *'If Lily eats three grapes, how many does Ralph eat?'*

Number of grapes that Lily eats	**Number of grapes that Ralph eats**
1	10
?	20
3	?

1:4

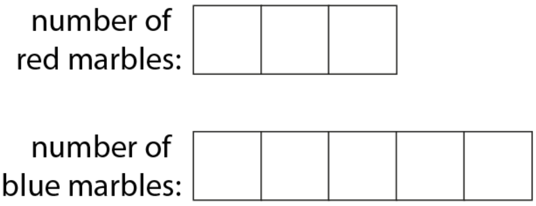
Now progress to examples where the ratio is *not* given in terms of 1:*x*. For example, *'Bijan has some marbles. For every five blue marbles, he has three red marbles. If Bijan has fifteen blue marbles, how many red marbles does he have? How many marbles does he have altogether?'*

Use the bar modelling approach first (*Method 1*, on the next page), to support children's understanding of the multiplicative relationships, then progress to using ratio grids (*Method 2*, on the next page). Children could use multilink cubes to represent the bars in *Method 1* (three cubes to represent the red marbles and five cubes to represent the blue marbles). Spend some time comparing the two methods, concluding that the reasoning and calculations are the same, but that the representations are different.

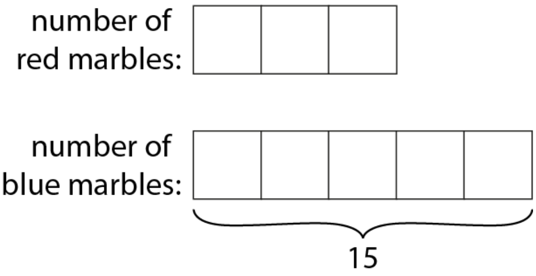
'Bijan has some marbles. For every five blue marbles, he has three red marbles. If Bijan has fifteen blue marbles, how many red marbles does he have? How many marbles does he have altogether?'

Method 1 – bar modelling:

- *'For every five blue marbles, there are three red marbles.'*

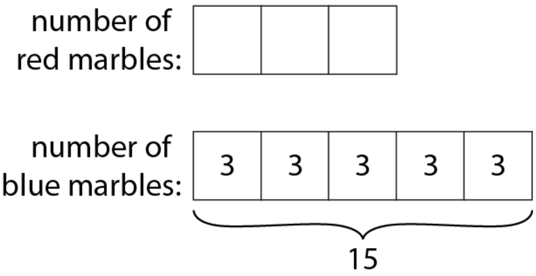


- *'There are fifteen blue marbles.'*



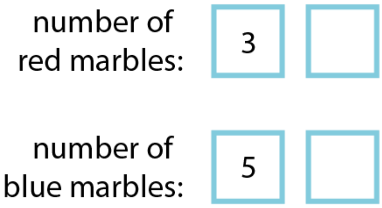
- *'Fifteen is divided into five units, so each unit has a value of <u>three</u>.'*

  15 ÷ 5 = **3** (and 5 × **3** = 15)

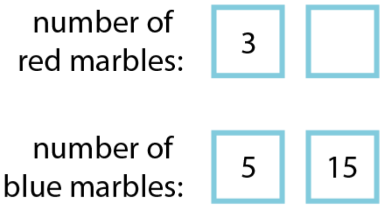


Method 2 – ratio grid:

- *'For every five blue marbles, there are three red marbles.'*

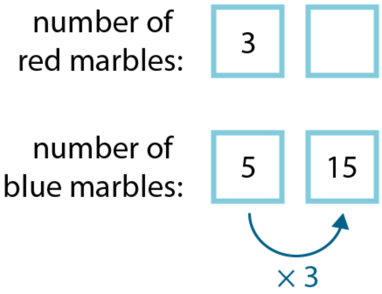


- *'There are fifteen blue marbles.'*



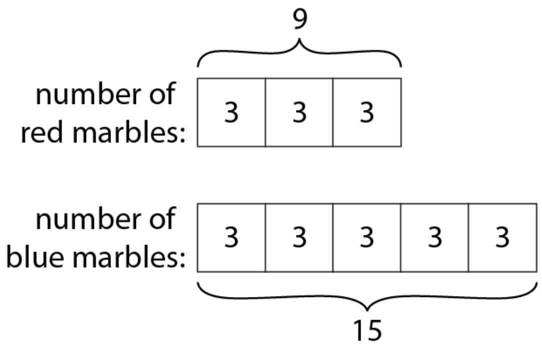
- *'To get from five to fifteen, I must <u>multiply by three</u>.'*

  5 × **3** = 15 (and 15 ÷ 5 = **3**)



- *'There are three units of red marbles and each unit has a value of <u>three</u>, so Bijan has nine red marbles.'*

$3 \times \mathbf{3} = 9$

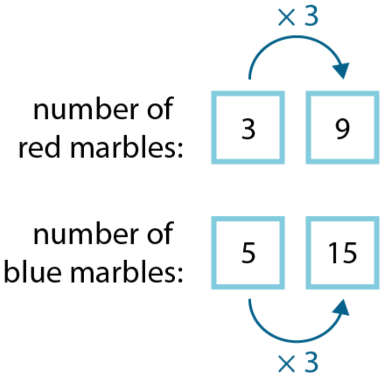


- *'Bijan has twenty-four marbles altogether.'*

$15 + 9 = 24$

- *'To get from three to the missing number, I must also <u>multiply by three</u>. Bijan has nine red marbles.'*

$3 \times \mathbf{3} = 9$



- *'Bijan has twenty-four marbles altogether.'*

$15 + 9 = 24$

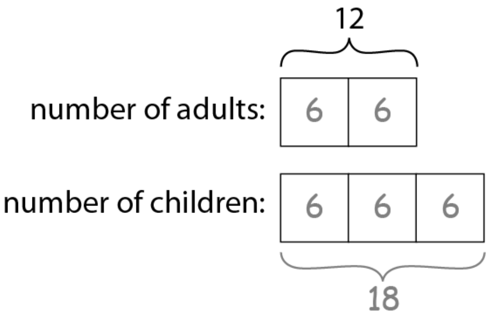
1:5

In the previous step, the problem gave the ratio between the two quantities and the absolute value of the *larger* quantity. Now, using the same methods, explore a problem where the ratio is given along with the absolute value of the *smaller* quantity, as exemplified and summarised opposite.

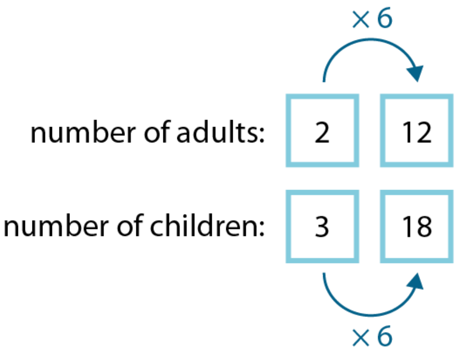
'Each team in a family quiz is made up of two adults and three children. If there are twelve adults in the competition:

- *how many children are there?*
- *how many people are there altogether (adults and children)?'*

Method 1 – bar modelling:



Method 2 – ratio grid:



Summary:

- *'There are eighteen children.'*

  $12 \div 2 \times 3 = 6 \times 3$
  $= 18$

- *'There are thirty people altogether.'*

  $18 + 12 = 30$

1:6

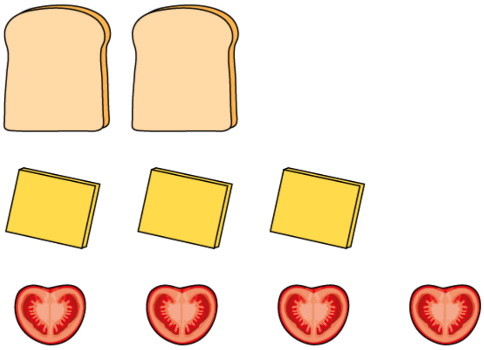
Now explore a problem where there are more than two variables. First provide the information about the ratios, for example, *'To make a cheese and tomato sandwich, we need two slices of bread, three slices of cheese and four slices of tomato.'* As in step *1:4*, encourage children to represent the relationships using multilink cubes (concrete) and the bar model (pictorial).

Then ask a range of different questions based on the ratios given, and use bar modelling to solve them; for example:

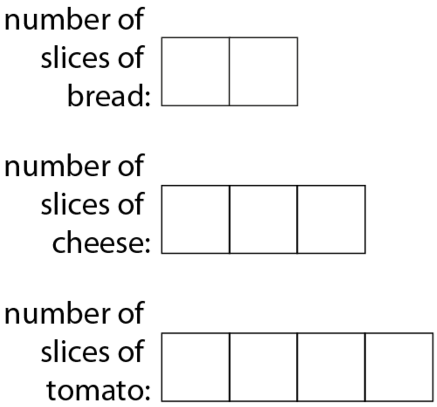
- *'How many slices each of bread, cheese and tomato do we need to make twelve sandwiches?'*
- *'If we use twenty-one slices of cheese, how many slices of tomato do we need?'*
- *'If a loaf of bread contains twenty-six slices, how many sandwiches can we make? What else will we need?'*

Then, alongside each of the bar model calculations, work with children to summarise the calculations using ratio grids.

'To make a cheese and tomato sandwich, we need two slices of bread, three slices of cheese and four slices of tomato.'



Bar model:



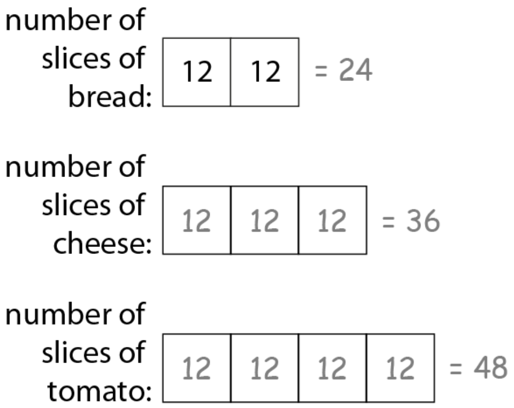
Ratio grid:

number of slices of bread: 2

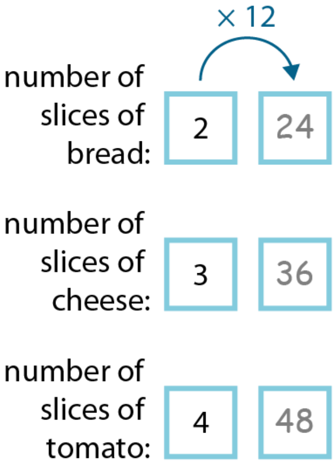
number of slices of cheese: 3

number of slices of tomato: 4

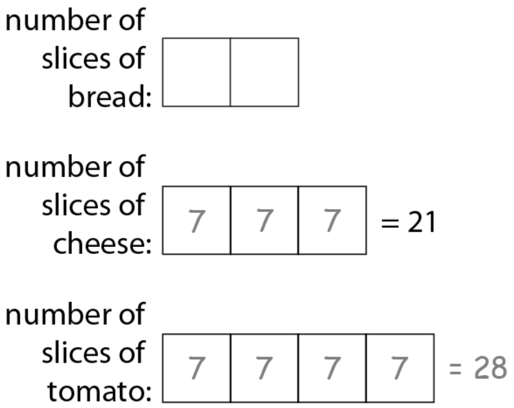
- *'How many slices each of bread, cheese and tomato do we need to make twelve sandwiches?'*



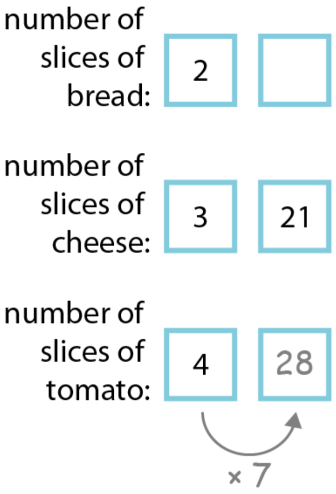
- *'For twelve sandwiches we need:'*
  - *'twenty-four slices of bread'*
    $2 \times 12 = 24$
  - *'thirty-six slices of cheese'*
    $3 \times 12 = 36$
  - *'forty-eight slices of tomato.'*
    $4 \times 12 = 48$



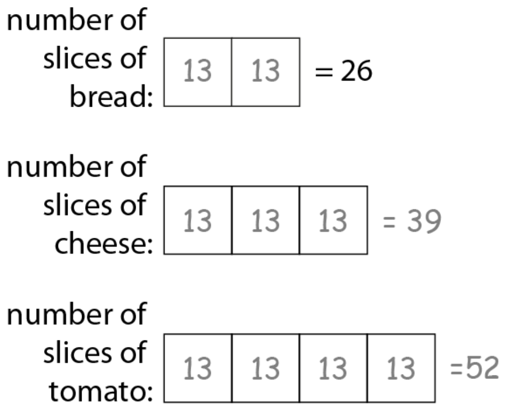
- *'If we use twenty-one slices of cheese, how many slices of tomato do we need?'*



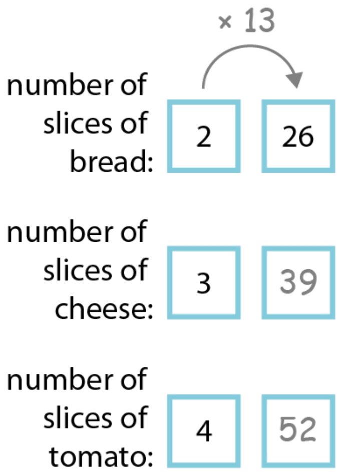
- *'We need twenty-eight slices of tomato.'*
  $21 \div 3 = 7$
  $4 \times 7 = 28$



- *‘If a loaf of bread contains twenty-six slices, how many sandwiches can we make? What else will we need?’*



- *‘If we have twenty-six slices of bread:’*
  - *‘we can make thirteen sandwiches’*
    26 ÷ 2 = 13
  - *‘we need thirty-nine slices of cheese’*
    3 × 13 = 39
  - *‘we need fifty-two slices of tomato.’*
    4 × 13 = 52



1:7 Explore a range of examples, as a class, using only the ratio-grid method. Include both discrete and continuous quantities. Ask questions that require children to think about the multiplicative relationships both ‘horizontally’ and ‘vertically’ in the ratio grid (these two possibilities are shown separately in the first part of *Example 1* on the next page, but are then represented together on a single ratio grid thereafter; for each problem, children only need to use one or the other to reason the answer). Also include problems that involve scaling to both larger quantities (as in *Example 2*, scaling the ingredients for two smoothies to find the ingredients for four or twenty smoothies) and to smaller quantities (as in *Example 2*, scaling the ingredients for two smoothies to find the ingredients for one smoothie).

For each example, begin by recording the information provided in the question. Then encourage children to identify and describe the relationships they can see between the different quantities; i.e., for the first question in *Example 1* on the next page:

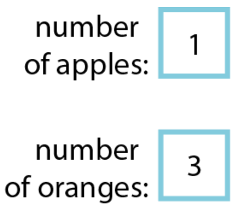
- *‘There are three times as many oranges as there are apples.’*
- *‘There are one-third as many apples as there are oranges.’*
- *‘Eight apples is eight times as many as one apple.’*

Children should then be able to use the relationships to answer the questions.

Note that a further row can be added to the ratio grid to show the total quantity of items (the sum of the preceding rows), as in *Example 1* on the next page.

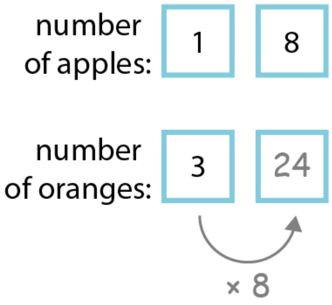
Example 1 – discrete quantities:

'Yukti is making bags of fruit. She puts one apple and three oranges in each bag.'



- *'If Yukti uses eight apples, how many oranges must she use?'*

Reasoning 'horizontally':

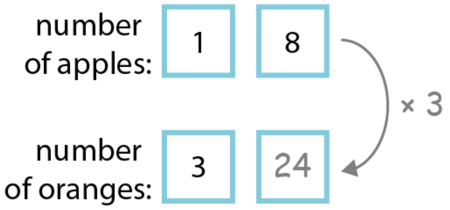


- *'To get from one to eight, I must* *<u>multiply by eight</u>.'*

  $1 \times \mathbf{8} = 8$ (and $8 \div 1 = \mathbf{8}$)
- *'To get from three to the missing number, I must also* *<u>multiply by eight</u>.'*

  $3 \times \mathbf{8} = 24$

Reasoning 'vertically':



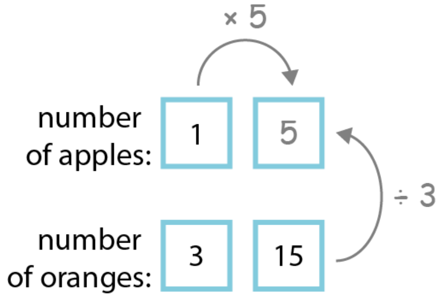
- *'To get from one to three, I must* *<u>multiply by three</u>.'*

  $1 \times \mathbf{3} = 3$ (and $3 \div 1 = \mathbf{3}$)
- *'To get from eight to the missing number, I must also* *<u>multiply by three</u>.'*

  $8 \times \mathbf{3} = 24$

- *'If Yukti uses eight apples, she must use twenty-four oranges.'*

- *'If Yukti uses fifteen oranges, how many apples must she use?'*



Reasoning 'horizontally'

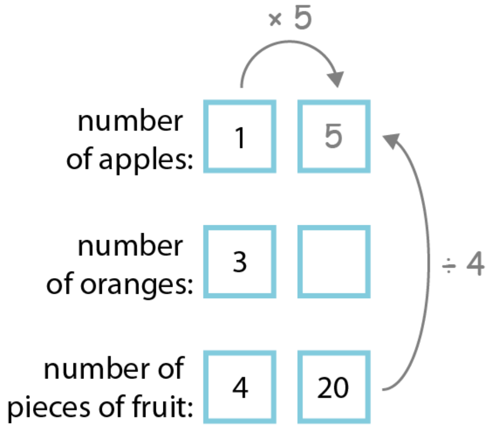
- *'To get from three to fifteen, I must <u>multiply by five</u>.'*
  $3 \times \mathbf{5} = 15$ (and $15 \div 3 = \mathbf{5}$)
- *'To get from one to the missing number, I must also <u>multiply by five</u>.'*
  $1 \times \mathbf{5} = 5$

Reasoning 'vertically'

- *'To get from three to one, I must <u>divide by three</u>.'*
  $3 \div \mathbf{3} = 1$
- *'To get from fifteen to the missing number, I must also <u>divide by three</u>.'*
  $15 \div \mathbf{3} = 5$

- *'If Yukti uses fifteen oranges, she must use five apples.'*

- *'How many apples would Yukti have used if she had used twenty pieces of fruit altogether?'*



Reasoning 'horizontally'

- *'To get from four to twenty, I must <u>multiply by five</u>.'*
  $4 \times \mathbf{5} = 20$ (and $20 \div 4 = \mathbf{5}$)
- *'To get from one to the missing number, I must also <u>multiply by five</u>.'*
  $1 \times \mathbf{5} = 5$

Reasoning 'vertically'

- *'To get from four to one, I must <u>divide by four</u>.'*
  $4 \div \mathbf{4} = 1$
- *'To get from twenty to the missing number, I must also <u>divide by four</u>.'*
  $20 \div \mathbf{4} = 5$

- *'If Yukti had used twenty pieces of fruit altogether, she would have used five apples.'*

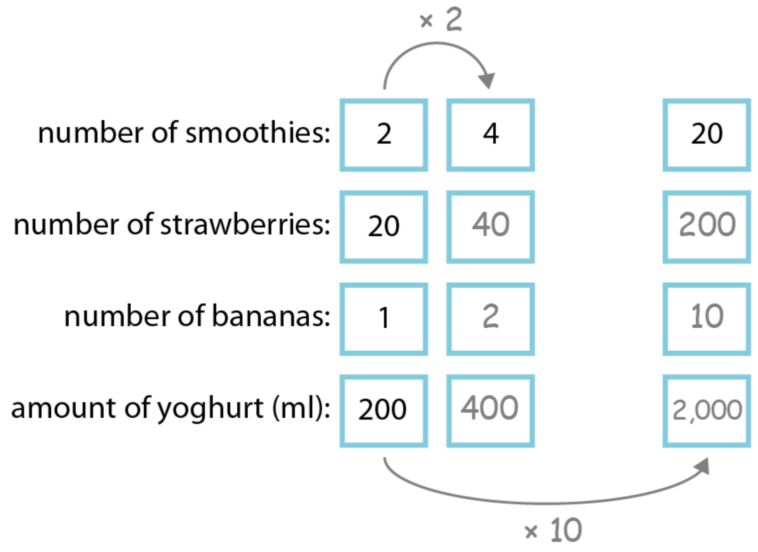
Example 2 – continuous measures:

Ingredients for two smoothies

- 20 strawberries
- 1 banana
- 200 ml yoghurt

number of smoothies:	2
number of strawberries:	20
number of bananas:	1
amount of yoghurt (ml):	200

- *'How much of each ingredient do you need to make:*
    - *four smoothies?*
    - *twenty smoothies?'*

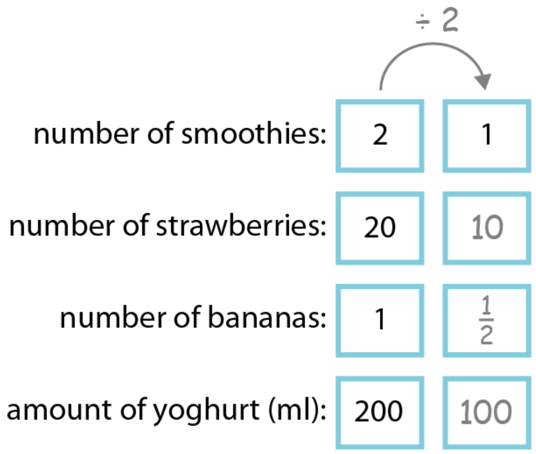


number of smoothies:	2	4	20
number of strawberries:	20	40	200
number of bananas:	1	2	10
amount of yoghurt (ml):	200	400	2,000

- *'To get from two smoothies to four smoothies, I must multiply by two.'*
- *So, I must multiply the amount of each ingredient by two.'*

- *'To get from two smoothies to twenty smoothies, I must multiply by ten.'*
- *'So, I must multiply the amount of each ingredient by ten.'*

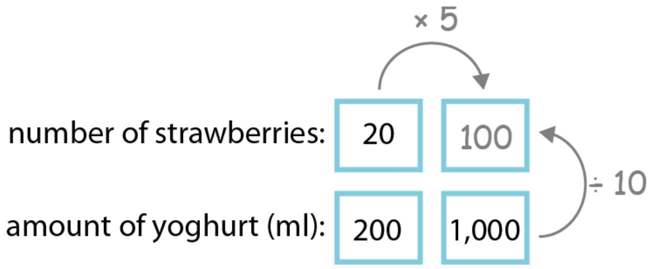
- *'How much of each ingredient do you need to make one smoothie?'*



number of smoothies:	2	1
number of strawberries:	20	10
number of bananas:	1	$\frac{1}{2}$
amount of yoghurt (ml):	200	100

- *'To get from two smoothies to one smoothie, I must divide by two.'*
- *'So, I must divide the amount of each ingredient by two.'*

- *'If you use one litre of yoghurt to make some smoothies, how many strawberries will you need?'*



number of strawberries:	20	100
amount of yoghurt (ml):	200	1,000

Reasoning 'horizontally'	Reasoning 'vertically'
- *'To get from two hundred to one thousand, I must multiply by five.'* 200 × **5** = 1,000 (and 1,000 ÷ 200 = **5**) - *'To get from twenty to the missing number, I must also multiply by five.'* 20 × **5** = 100	- *'To get from two hundred to twenty, I must divide by ten.'* 200 ÷ **10** = 20 - *'To get from one thousand to the missing number, I must also divide by ten.'* 1,000 ÷ **10** = 100

Note for teachers: this example illustrates why it is important not to include the units 'ml' inside the boxes, since *1,000 ml ÷ 10* is not equal to 100 (but to 100 ml).

1:8

Provide children with practice solving ratio problems, including those:

- where the ratio is in the form 1:*x* (e.g. 1:3 or 1:5*)*
- where the ratio is in the form *x*:*y* ($x \neq 1$; $y \neq 1$) (e.g. 2:3 or 4:7)
- with more the two variables.

Include both cardinal and measures contexts.

Example problems:

- *'Nicky makes some fruit juice. For every one orange, she uses four strawberries. If she uses nine oranges how many strawberries does she use?'*
- *'Some children are planting trees. The children are put into groups of eight, and each group is given three trees to plant.'*
  - *'If there are thirty-two children, how many trees will be planted?'*
  - *'If eighteen trees are planted, how many children are there?'*
- *'A shop sells packs of stationery items. One pack contains four pens and two notepads.'*
  - *'If I buy seven packs, how many pens will I have?'*
  - *'If I have ten notepads, how many pens do I have?'*
  - *'If I have twelve pens, how many notepads do I have?'*

'The table shows a set of equipment needed to play a game of rounders.'

Item	Number
Balls	1
Bats	2
Posts*	6

*bases plus bowler and batter positions

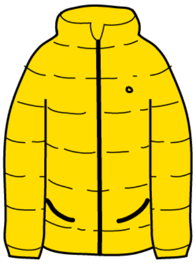
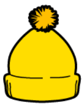
'Mrs Hopper buys some sets of rounders equipment. She has forty-five items altogether.'

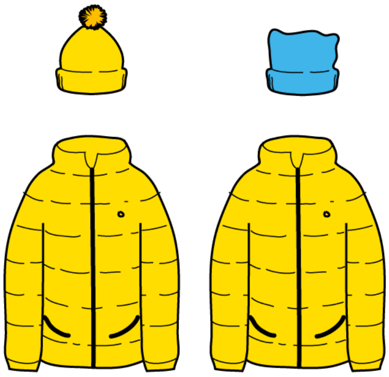
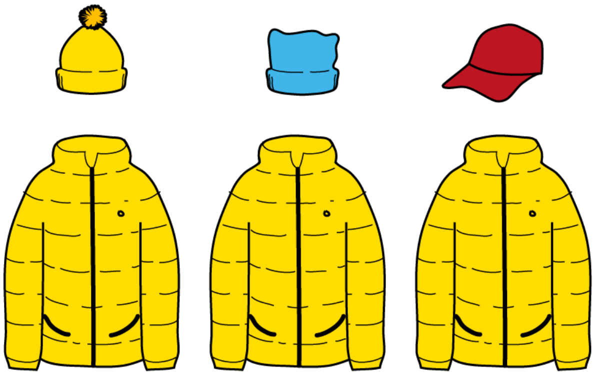
- *'How many sets of rounders of equipment has she bought?'*
- *'How many bats are there?'*

Teaching point 2:

Multiplication and understanding of correspondence can be used to calculate the number of possible combinations of items.

Steps in learning

	Guidance	**Representations**
2:1	In this teaching point, children explore correspondence problems in the context of calculating the number of possible combinations of certain items, such as *'If Megan has four coats and three hats, how many different outfits can she make?'* Begin with one hat and one coat, and establish that there is only one possible outfit/combination. Note that we are making the assumption that Megan must wear one coat and one hat; just wearing the coat, or just wearing the hat, do not count as another two outfits.	*'If Megan has one coat and one hat, how many different outfits can she make?'*  
2:2	Now increase the number of hats that Megan has, recording the number of possible outfits/combinations in a table. To connect to children's prior learning, and to prepare for the upcoming steps, use the stem sentence from step *1:1*: ***'For every ___ ___, there are ___ ___.'*** Referring to the table, ask children what the relationship is between the number of hats, coats and outfits, encouraging them to notice the multiplicative relationship.	

One coat, two hats:	One coat, three hats:
Coats: 1 	Coats: 1 
Hats: 2  	Hats: 3   
Combinations/outfits: 2 	Combinations/outfits: 3 
- *'For every one coat, there are two possible hats.'* - *'There is one coat, so there are two outfits.'* $1 \times 2 = 2$	- *'For every one coat, there are three possible hats.'* - *'There is one coat, so there are three outfits.'* $1 \times 3 = 3$

Summary table:

Number of coats	Number of hats	Combinations
1	1	1
1	2	2
1	3	3

$1 \times 1 = 1$
$1 \times 2 = 2$
$1 \times 3 = 3$

number of **coats** × number of **hats** = number of **outfits**

2:3

Now, building on the final case from step *2:2* (one coat and three hats), add another coat. Work systematically to find the number of possible outfits. Continue to use the stem sentence:

'For every ___ ___, there are ___ ___.'

to support children in understanding the connection between the context and multiplicative reasoning.

Add this new example to the summary table, then work through several more examples, each time working systematically, and making the connection to multiplication.

When the pattern becomes clear, you could take another example, using multiplication to calculate the number of possibilities before arranging the items to verify the answer.

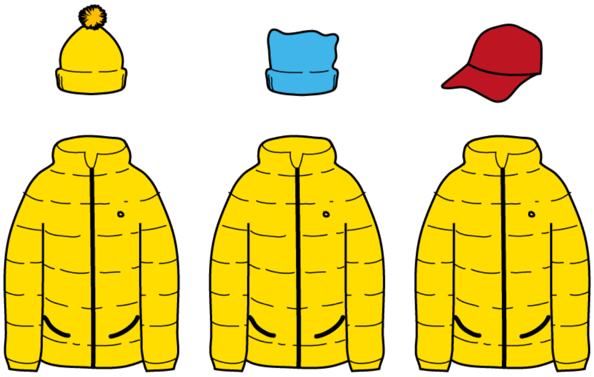
'If Megan has two coats and three hats, how many different outfits can she make?'

Coats: 2

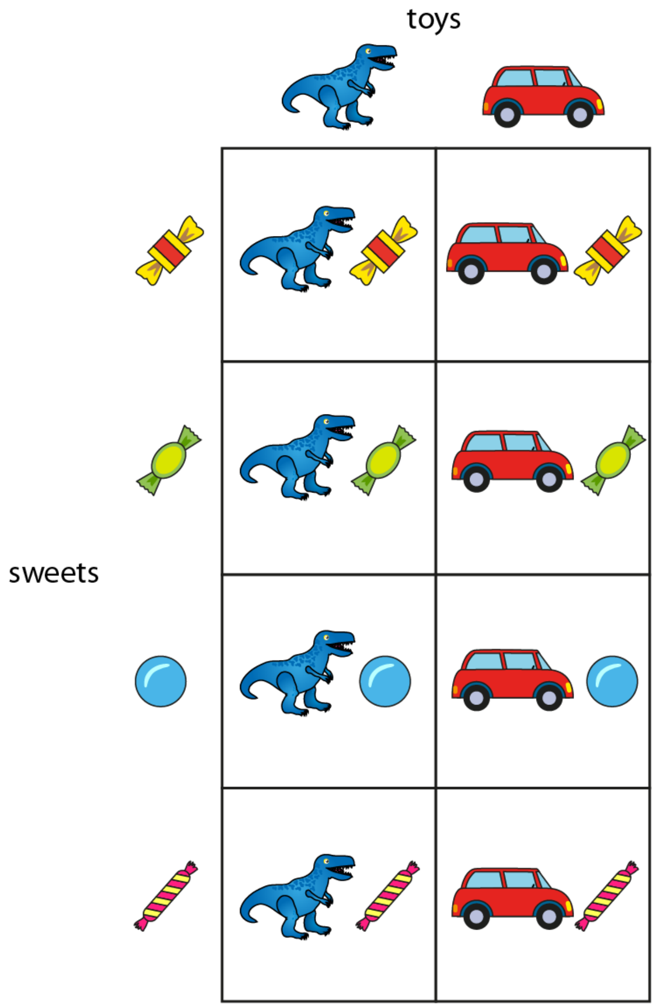
Hats: 3

Combinations/outfits: 6

- *'If Megan chooses the yellow coat, there are three possible hats.'*



- *'If Megan chooses the blue coat, there are three possible hats.'*

		• *'For every one coat, there are three possible hats.'* • *'There are two coats, so there are six outfits.'* 2 × 3 = 6 number of **coats** × number of **hats** = number of **outfits**
2:4	Work through another context. You can illustrate how the combinations can be connected to children's understanding of arrays, as shown opposite.	*'Emma is making party bags. In each bag she puts a toy and a sweet. How many different ways can she create a party bag if she has two types of toy and four types of sweet?'*  • *'For every one toy, there are four possible sweets.'* • *'There are two toys, so there are eight possible party bags.'* 2 × 4 = 8 number of types of **toys** × number of types of **sweets** = number of types of **party bags**

2:5	Provide children with practice solving different correspondence problems involving calculating combinations, for example:
	• *'Rajesh takes seven different pairs of socks and three different pairs of shoes on holiday with him. How many different combinations of shoes and socks does he have?'*
	• *'Before a football game, each person on team A shakes hands with each person on team B. How many handshakes are there if there are:*
	  • *five people in each team?*
	  • *eleven people in each team?'*
	• Dòng nǎo jīn
	  • *'Frankie has some different pairs of trousers and six different T-shirts. He can make twenty-four different outfits. How many pairs of trousers does Frankie have?'*
	  • *'I have two boxes, each containing a set of different toys. There is only one of each type of toy. I draw one toy out of each box to make a pair. If I can make twenty-four different pairs of toys, how many toys were in each box?'*
	When writing problems, make sure that the items in each of the two sets differ, otherwise the number of combinations will be less than the product of the quantities of items in the two sets.	

Teaching point 3:

Scaling can be used to make and interpret maps.

Steps in learning

	Guidance	**Representations**
3:1	See below	See below

Guidance

This teaching point links to what children have already learnt about scaling lengths (segment *2.17 Structures: using measures and comparison to understand scaling*), and 1:*x* and *x*:*y* multiplicative relationships.

Begin by showing a map with a simple scale, e.g. 1 cm:2 km. Ask children what they think the scale means, and use the following sentence: '*Every one centimetre on the map represents two kilometres on the island.*'

Spend some time exploring distances on the map, beginning by converting from map distances to real-world distances, for example:

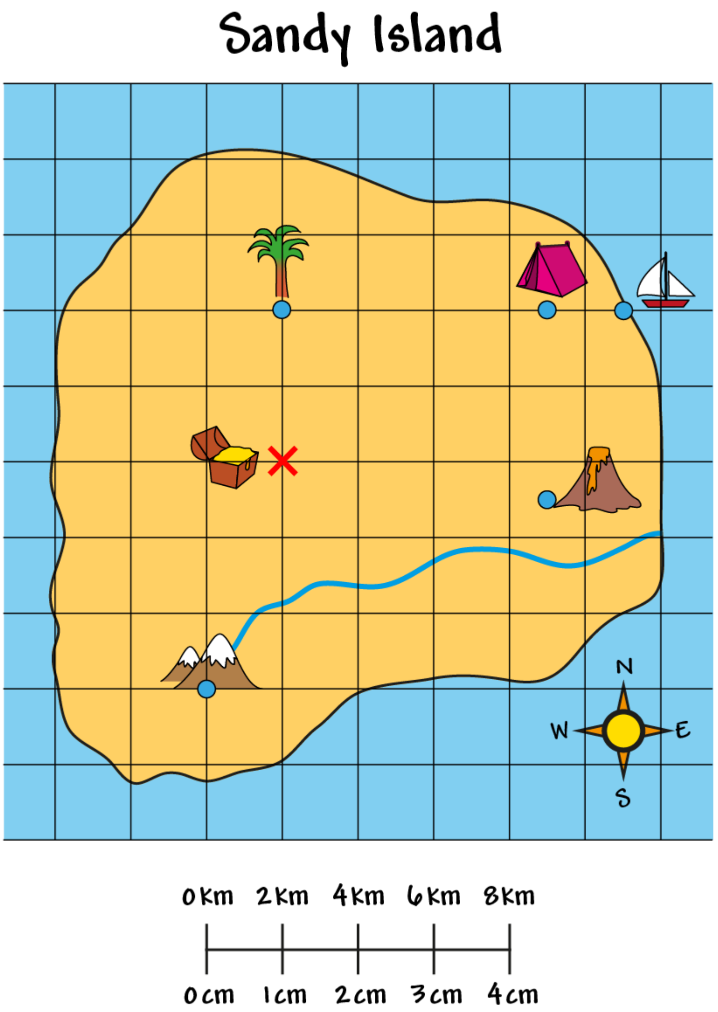
- *'Roughly how wide is the island?'*
- *'How far is it from the tree to the treasure?'*

As you work through the questions, extend the scale line (double number line) to represent the distances. Also include some distances that fall between the marked intervals, for example: '*How far is it from the camp to the tree?*' (3.5 cm/7 km)

Then progress to converting real-world distances to map distances, for example:

- *'What does one kilometre look like on the map?'*
- *'There is a pond four kilometres from the tree. What would this distance be on the map?'*

Representations

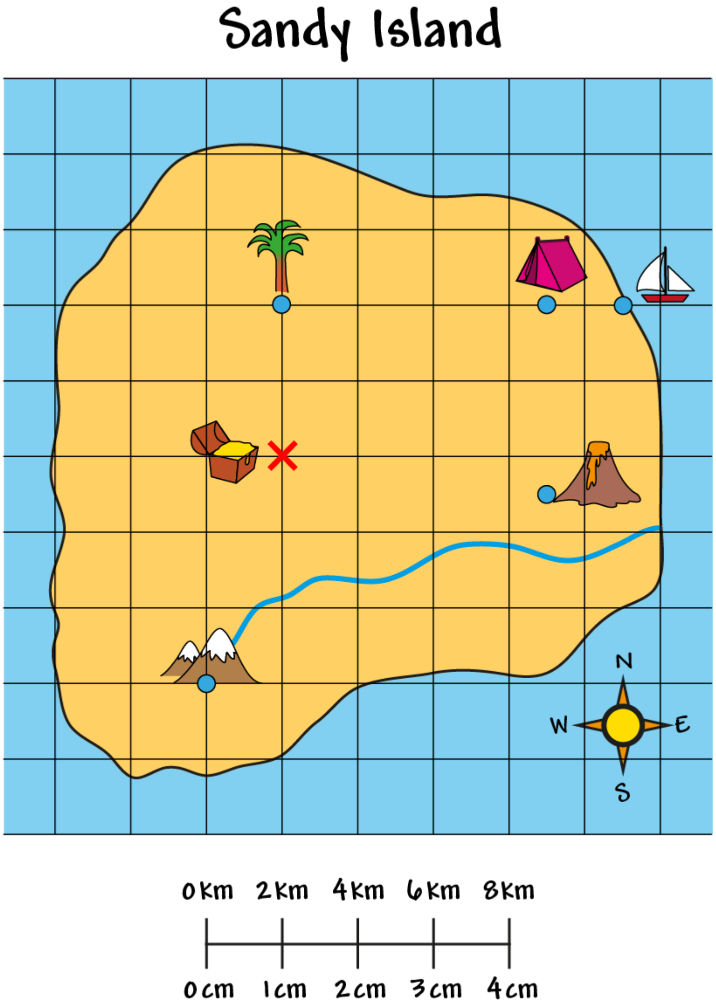


3:2	Now spend some time creating different scale lines (double number lines) for a variety of scales, for example: • 1 cm:4 km • 2 cm:3 km For each scale, ask children: • *'If I know this, what else do I know?'* • to draw their own double number line to compare distances up to 10 cm.	
3:3	Set children the task of drawing their own scale map, using centimetre squared paper. For the first example, provide children with some distances between features, and a scale to use, for example: • *'The island is 21 km from east to west, and 15 km from north to south.'* • *'There is a camp 3 km from the coast.'* • *'There is a mountain 9 km from the camp.'* • *'The treasure is 4.5 km from the mountain.'* • *'Use the following scale: 1 cm represents 3 km.'* Encourage children to draw a scale line (double number line) covering the required distances before they start their drawing. Then provide a second set of distances/features and ask children to work out a suitable scale for themselves.	
3:4	Now return to the treasure map from step *3:1*. Introduce the idea of another island 40 km to the west of Sandy Island, and ask pupils to consider how far away this would be on the map. (It wouldn't be sensible to actually draw this at the same scale, but children should be able to calculate where it would be.) Extending the scale line (double number line) up to 40 km would be cumbersome (and would become impossible as distances increase further). Remind children of the ratio-grid method used in *Teaching point 1*, and model how this can be used to calculate what distance 40 km would be on the map.	

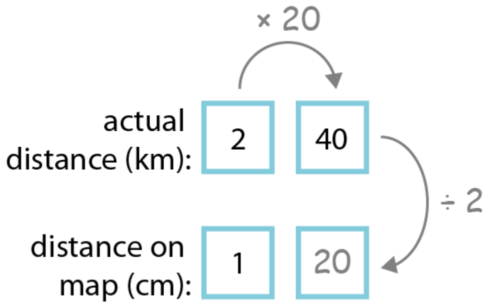
As noted in step *1:7*, it is important *not* to include the units inside the boxes of the ratio-grid; the multiplicative relationships being used are between the abstract numbers, and not the actual measures. For example:

- 40 ÷ 2 = 20
  So, 40 km in real life would be drawn as 20 cm on the map. ✓
- 40 km ÷ 2 = 20 cm ✗

Explore some other distances in this way, first based on the treasure map from step *3:1*, and then using different ratios. The calculations can be 'reasoned' in more than one way (for the examples below, both methods are shown). Sometimes the arithmetic is much simpler for one of the ways of reasoning – encourage children to notice when this is the case.



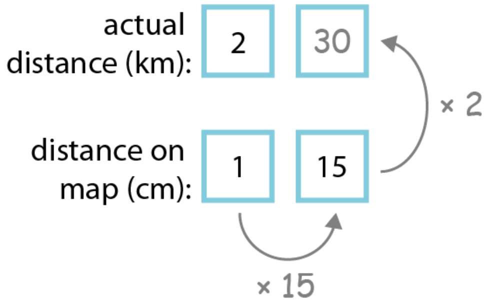
- *'Rocky Island is 40 km west of Sandy Island. What would this distance be on the map?'*



Reasoning 'horizontally'	Reasoning 'vertically'
- *'To get from two to forty, I must multiply by twenty.'* $2 \times \mathbf{20} = 40$ (and $40 \div 2 = \mathbf{20}$) - *'To get from one to the missing number, I must also multiply by twenty.'* $1 \times \mathbf{20} = 20$	- *'To get from two to one, I must divide by two.'* $2 \div \mathbf{2} = 1$ - *'To get from forty to the missing number, I must also divide by two.'* $40 \div \mathbf{2} = 20$

- *'On the map, Rocky Island would be 20 cm away from Sandy Island.'*

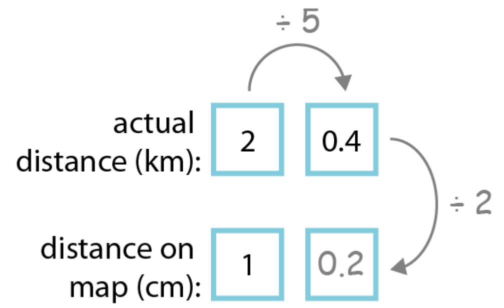
- *'What would 15 cm on the map represent in real life?'*



Reasoning 'horizontally'	Reasoning 'vertically'
- *'To get from one to fifteen, I must multiply by fifteen.'* $1 \times \mathbf{15} = 15$ (and $15 \div 1 = \mathbf{15}$) - *'To get from two to the missing number, I must also multiply by fifteen.'* $2 \times \mathbf{15} = 30$	- *'To get from one to two, I must multiply by two.'* $1 \times \mathbf{2} = 2$ (and $2 \div 1 = \mathbf{2}$) - *'To get from fifteen to the missing number, I must also multiply by two.'* $15 \times \mathbf{2} = 30$

- *'Fifteen centimetres on the map would represent thirty kilometres in real life.'*

- *'There is a pond 0.4 km away from the camp. How far would this be on the map?'*

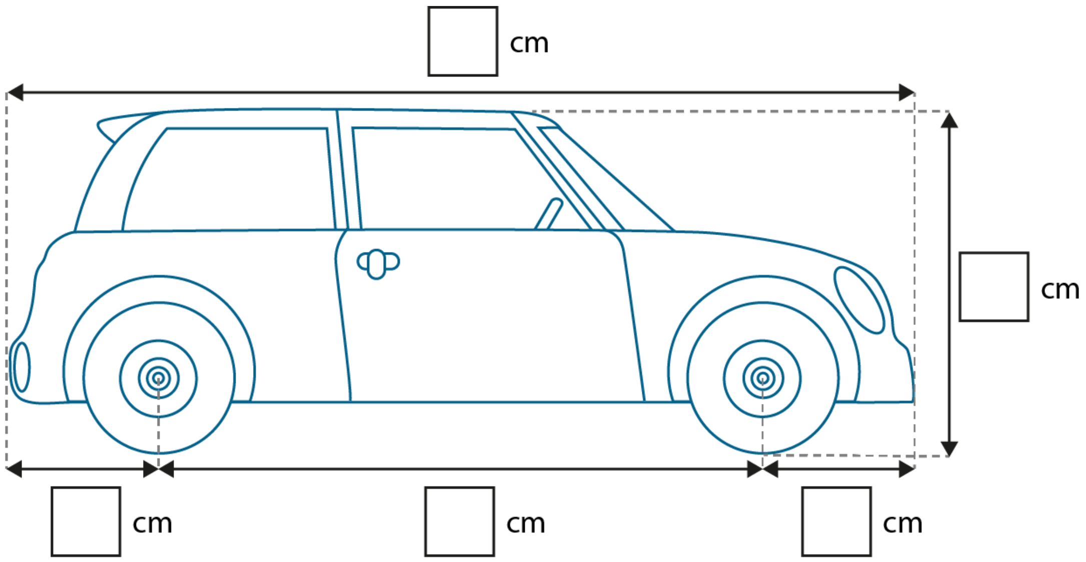


Reasoning 'horizontally'	Reasoning 'vertically'
• *'To get from two to zero-point-four, I must divide by five.'* 2 ÷ **5** = 0.4 • *'To get from one to the missing number, I must also divide by five.'* 1 ÷ **5** = 0.2	• *'To get from two to one, I must divide by two.'* 2 ÷ **2** = 1 • *'To get from zero-point-four to the missing number, I must also divide by two.'* 0.4 ÷ **2** = 0.2

- *'On the map, the pond would be 0.2 cm away from the camp.'*

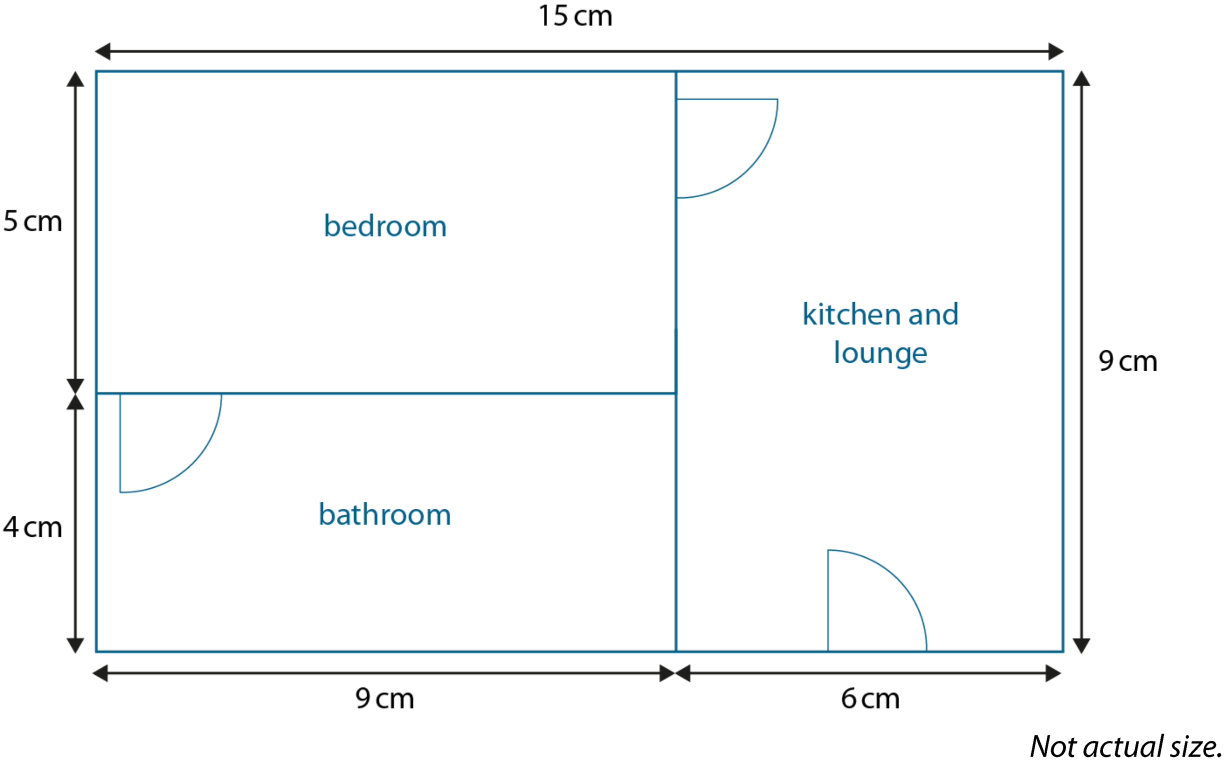
3:5 Provide children with practice interpreting a range of different scales, including real maps. You can also widen the scope of contexts to other scale drawings and include larger numbers in the calculations, as exemplified below.

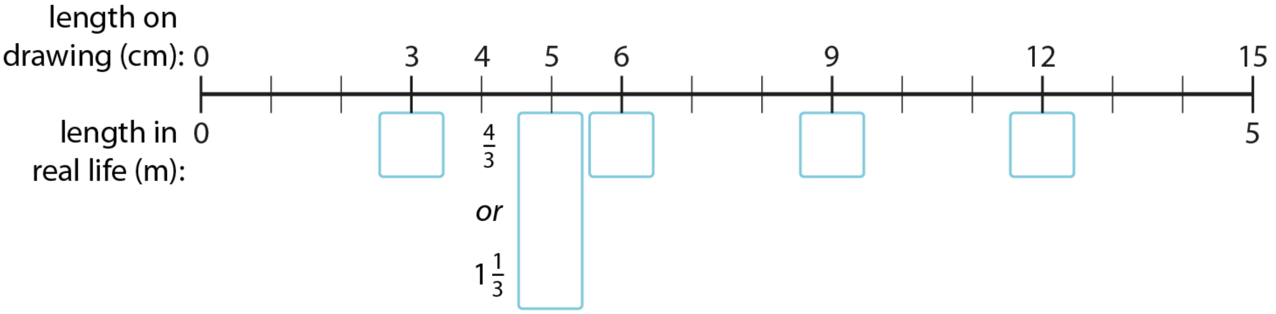
'1 cm on this drawing represents 30 cm in real life. Fill in the real-life measurements of the car.'



3:6 Dòng nǎo jīn:

'The drawing shows an apartment. 15 cm on this drawing represents 5 m in real life. Fill in the missing numbers on the number line to help you calculate the real measurements of each of the rooms. Represent any measurements that are not whole numbers as fractions.'





dimensions of apartment: ________________

dimensions of kitchen/lounge: ________________

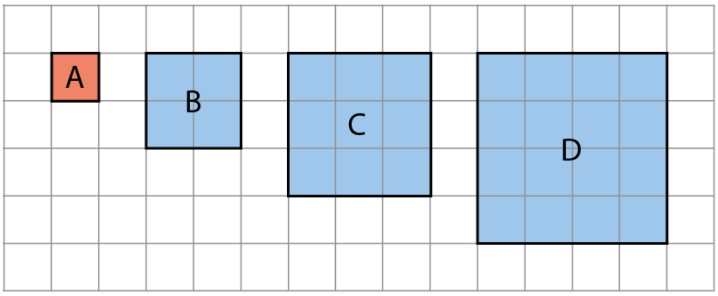
dimensions of bathroom: ________________

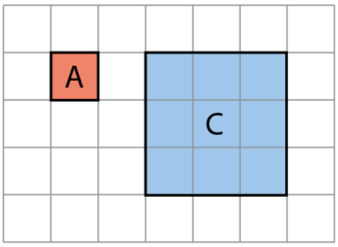
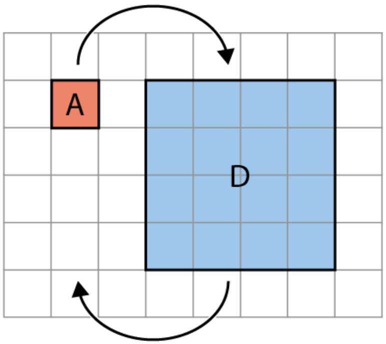
dimensions of bedroom: ________________

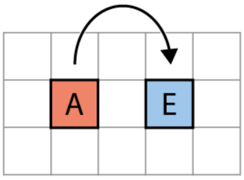
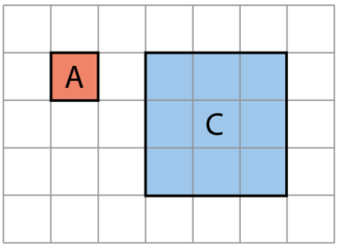
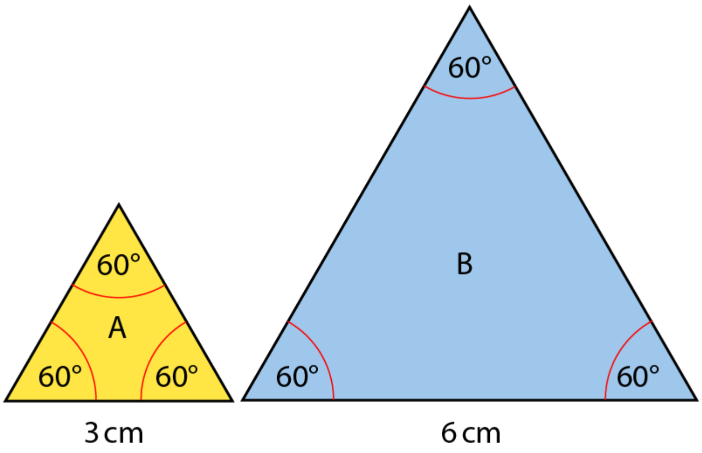
Teaching point 4:

There is a proportional relationship between the dimensions of similar shapes; if the scale factor and the dimensions of one of the shapes is known, the dimensions of the similar shape can be calculated; if the dimensions of both of the shapes are known, the scale factor can be calculated.

Steps in learning

	Guidance	**Representations**
4:1	Two shapes are *'congruent'* if they can be transformed into one another by reflection, translation or rotation only (or any combination of these). Two shapes are *'similar'* if they can be transformed into one another by scaling (there may also be a reflection, translation or rotation). In this teaching point, children will work with similar shapes, using proportional reasoning to solve problems about scaling the side-lengths. Begin by exploring squares, since it is easy to see that squares of different side-lengths are similar shapes. For now, keep the shapes in the same orientation. Show a selection of different-sized squares. Each of the larger squares should be related to the smallest square by a whole-number scale factor. Ask children to compare the smallest square with each of the larger squares by comparing the length of one of the sides, using the stem sentence: ***'The length of one of the sides of square is ___ times the length of one of the sides of square ___.'*** This could be simplified, over time, to: ***'The side-length of square ___ is ___ times the side-length of square ___.'*** It is important to refer to the dimensions (side-length); avoid saying, for example, *'square B is two times the size of square A'*, since 'size' is an imprecise term and could refer to the area of the squares which are related by	 Example comparison: - *'The length of one of the sides of square B is two times the length of one of the sides of square A.'* side-length of B = side-length of A × **2** - *'The length of one of the sides of square A is one-half times the length of one of the sides of square B.'* side-length of A = side-length of B × $\mathbf{\frac{1}{2}}$

	a different scale factor than the dimensions. Ensure that children can describe the larger squares in terms of the smaller square and vice versa, as exemplified on the previous page, and can write multiplication equations to represent the relationships. Also ensure children's attention is drawn to the *multiplicative* relationship between side-lengths and not to the *additive* relationship.	
4:2	Now describe the relationships using the term *'scale factor'*. Begin with an example from the previous step that involves enlargement (e.g. A → C), repeating the stem sentence comparing the larger square to the smaller square: *'The length of one of the sides of square C is three times the length of one of the sides of square A.'* Then model use of the *term 'scale factor'*, using the stem sentence: ***'To change shape ___ into shape ___, scale the side-lengths by a scale factor of ___.'*** Connect the term *'scale factor'* to the multiplier in the multiplication equation. Then, in the same way, describe the scale factor for the corresponding reduction (C → A).	 • *'To change shape A into shape C, scale the side-lengths by a scale factor of three.'* side-length of C = side-length of A × **3** • *'To change shape C into shape A, scale the side-lengths by a scale factor of one-third.'* side-length of A = side-length of C × $\mathbf{\frac{1}{3}}$
4:3	Compare some of the other squares in the same way as in step *4:2*, including comparing a square to itself (scale factor of '1'), then generalise: • ***'If the scale factor is greater than one, the shape is made larger. We can say the shape is enlarged.'*** • ***'If the scale factor is equal to one, the shape is the same size.'*** • ***'If the scale factor is less than one, the shape is made smaller. We can say the shape is reduced.'*** Note that, at Key Stage 4, the term *'enlarge'* is used for both an increase	side-length × 4 **enlarged**  side-length × $\frac{1}{4}$ **reduced**

	and decrease in size (i.e. it is applied to cases with scale factors both smaller than and greater than one). At this stage, it is recommended that children focus on the phrases 'made larger' and 'made smaller'.	side-length × 1 **unchanged** 
4:4	Now introduce the term 'ratio' to describe the relationship between the dimensions of shapes. Revisit the example from step *4:2*, summarising the scale factor, then describing the relationship between the dimensions using the following stem sentence: ***'The ratio of the dimensions of shape ___ to the dimensions of shape ___ is equal to ___-to-___.'***	 - *'To change shape A into shape C, scale the side-lengths by a scale factor of three.'* - *'The ratio of the dimensions of shape A to the dimensions of shape C is equal to one-to-three.'* - *'We can write this as:'* dimensions of A : dimensions of C = 1 : 3 - *'To change shape C into shape A, scale the side-lengths by a scale factor of one third.'* - *'The ratio of the dimensions of shape C to the dimensions of shape A is equal to three-to-one.'* - *'We can write this as:'* dimensions of C : dimensions of A = 3 : 1
4:5	Compare some other pairs of similar regular polygons, such as equilateral triangles and regular hexagons. Include situations where the smaller shape does not have a side-length of one unit so that children have to work out the scale factor; also include measurements in centimetres (or metres) rather than just 'unit squares' that have been used so far. Then examine the examples explored so far. Explain that, when a shape has been enlarged or reduced, we say that the original shape and the new shape are *'similar'*; similar shapes have the same name. Draw attention to the fact that corresponding sides are	Equilateral triangles: 

proportional, and corresponding angles are equal; i.e. in the example opposite, draw attention to the fact that:

- the internal angles of triangle B are the same as the internal angles of triangle A (60°)
- to get from A to B, each of the side-lengths has been scaled by the same scale factor.

- *'The triangles are similar.'*

Transforming A to B:

6 cm = 3 cm × **2**

side-length of B = side-length of A × **2**

- *'To change triangle A into triangle B, scale the side-lengths by a scale factor of two.'*
- *'The ratio of the dimensions of triangle A to the dimensions of triangle B is equal to one-to-two.'*
- *'We can write this as:'*

dimensions of A : dimensions of B = 1 : 2

Transforming B to A:

$3\text{ cm} = 6\text{ cm} \times \mathbf{\frac{1}{2}}$

side-length of B = side-length of A × $\mathbf{\frac{1}{2}}$

- *'To change triangle B into triangle A, scale the side-lengths by a scale factor of one-half.'*
- *'The ratio of the dimensions of triangle B to the dimensions of triangle A is equal to two-to-one.'*
- *'We can write this as:'*

dimensions of B : dimensions of A = 2 : 1

4:6

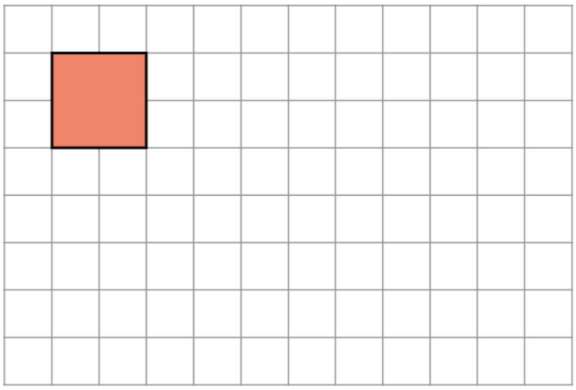
At this point, give children practice working with similar regular shapes, including:

- working out scale factors from given side-lengths
- working out the side-lengths of an enlarged or reduced shape
- drawing a new shape given the scale factor and the original side-lengths.

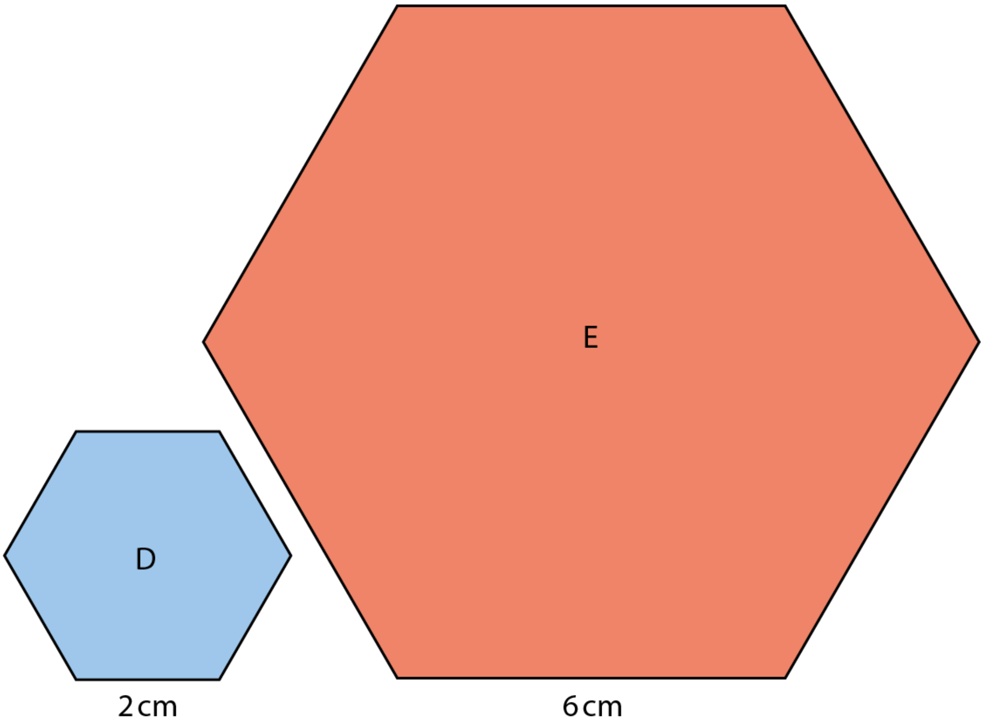
You may want to use squared paper to facilitate comparison and drawing.

Drawing similar shapes:

'Draw the shape that is produced by scaling the lengths of the sides of this square by a scale factor of two.'



Working out scale factors and side-lengths:

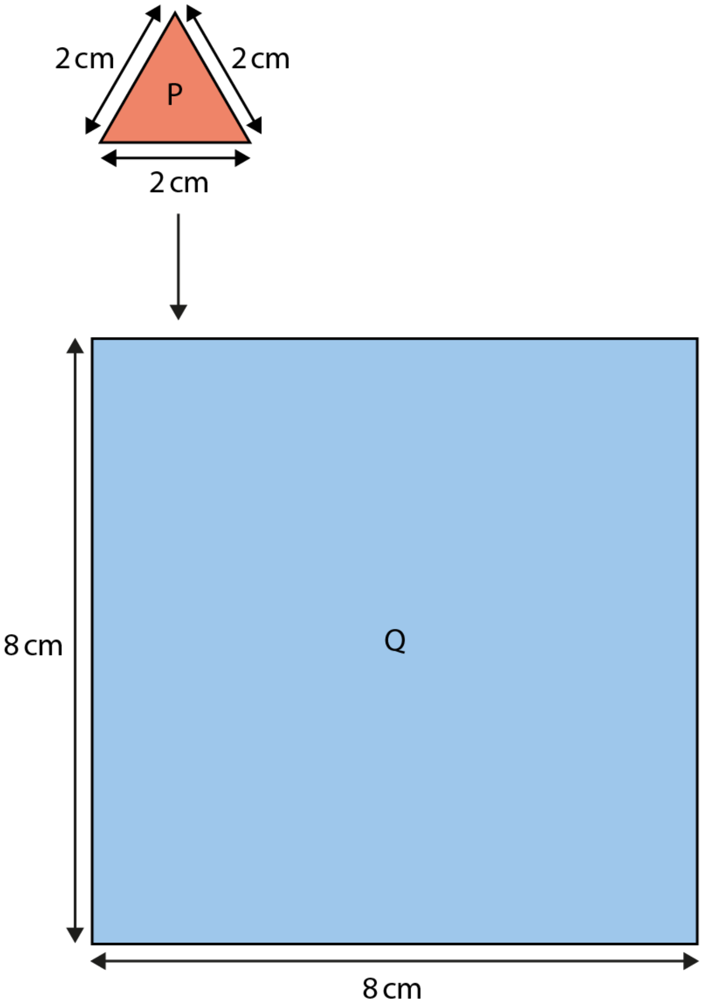


- *'What scale factor is used to change the side-lengths of hexagon D into the side-lengths of hexagon E?'*
- *'What scale factor is used to change the side-lengths of hexagon E into the side-lengths of hexagon D?'*
- *'A third hexagon, F, can be drawn by scaling the side-lengths of hexagon D by a scale factor of three. What is the length of the sides of hexagon F?'*
- *'Fill in the missing numbers.'*

side-length of D : side-length of F = ☐ : ☐

True/false-style problem:

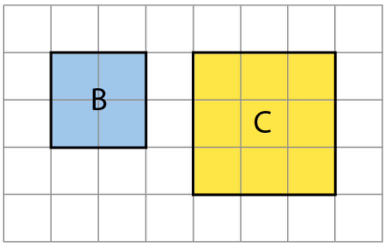
'Nicky says that shape P has been scaled by a factor of four to make shape Q, because the side-length of shape P is two centimetres and the side-length of shape Q is four times the size. Do you agree or disagree? Explain your answer.'



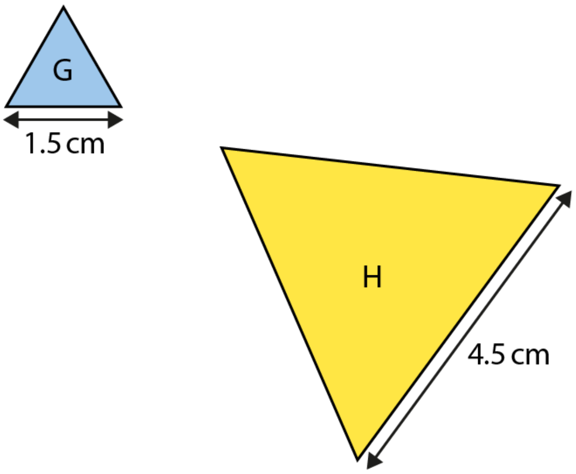
Dòng nǎo jīn:

'Express your answers to these questions as fractions.'

- *'What scale factor is used to change the side-lengths of square B into the side-lengths of square C?'*
- *'What scale factor is used to change the side-lengths of square C into the side-lengths of square B?'*



- *'What scale factor is used to change the side-lengths of equilateral triangle G into equilateral triangle H?'*



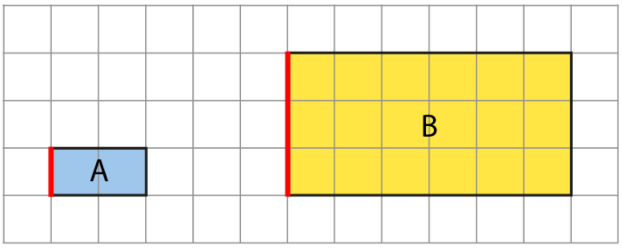
4:7

Now extend to irregular polygons, beginning with irregular rectangles. The language and mathematics are the same as for the squares in steps *4:1–4:4* (and the other regular polygons in earlier steps), but now children must ensure they are comparing the correct dimensions (corresponding side-lengths). Amend the language accordingly: ***'To change shape ___ into shape ___, scale the dimensions by a scale factor of ___.'***

Continue to also use the term 'ratio' to describe the relationship between the shapes: '***The ratio of the dimensions of shape ___ to the dimensions of shape ___ is equal to ___-to-___.'***

Example 1:

- Comparing the short sides (heights)



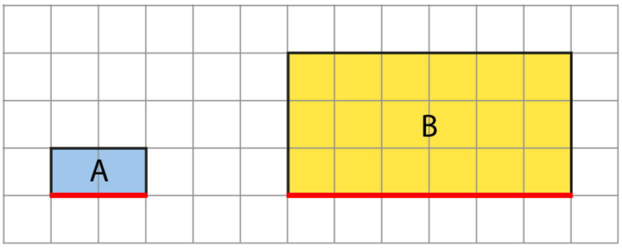
$3 = 1 \times \mathbf{3}$

height of B = height of A × **3**

Work with a variety of pairs of similar rectangles, including those shown at different orientations, to ensure that children can correctly compare corresponding side-lengths.

You could use Cuisenaire® rods as a means to explore the relationship between dimensions. For example, if rectangle A (opposite) had dimensions equivalent to one pink rod (short side) by one tan rod (long side), rectangle B would have dimensions equivalent to three pink rods by three tan rods. When using Cuisenaire® rods in this way, avoid assigning a value to the length of a rod (i.e. do not say that pink = 4 and tan = 8); instead, focus on the proportional relationship.

- Comparing the long sides (widths)



$6 = 2 \times \mathbf{3}$

width of B = width of A × **3**

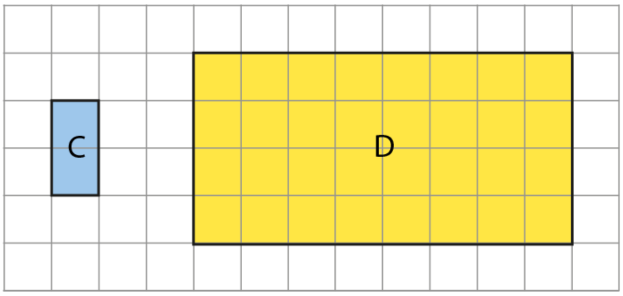
- Comparing rectangles A and B
  - *'The rectangles are similar because both side-lengths have been scaled by the same scale factor.'*
  - *'To change rectangle A into rectangle B, scale the dimensions by a scale factor of three.'*
  - *'The ratio of the dimensions of rectangle A to the dimensions of rectangle B is equal to one-to-three.'*

  dimensions of A : dimensions of B = 1 : 3

  - *'To change rectangle B into rectangle A, scale the dimensions by a scale factor of one-third.'*
  - *'The ratio of the dimensions of rectangle B to the dimensions of rectangle A is equal to three-to-one.'*

  dimensions of B : dimensions of A = 3 : 1

Example 2:



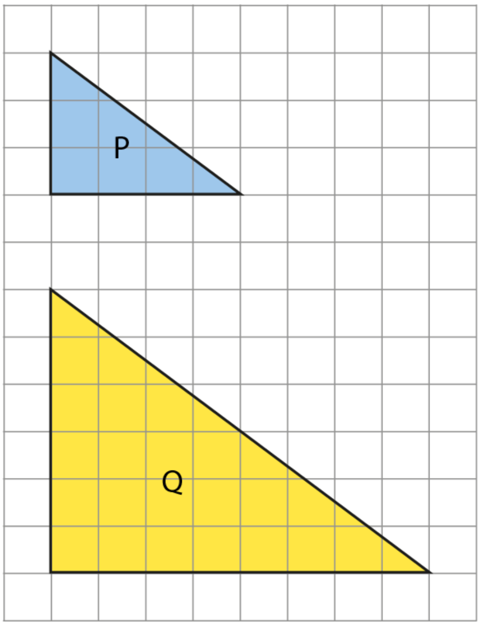
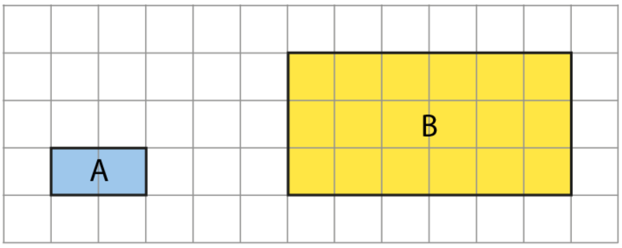
- Comparing the dimensions

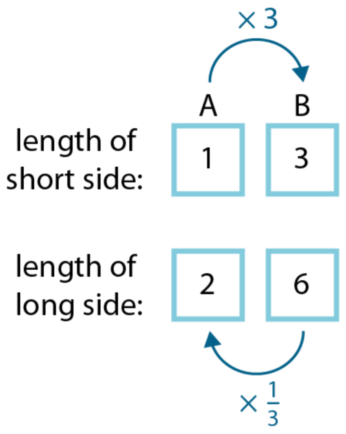
$4 = 1 \times \mathbf{4}$

short dimension of D = short dimension of C × **4**

$8 = 2 \times \mathbf{4}$

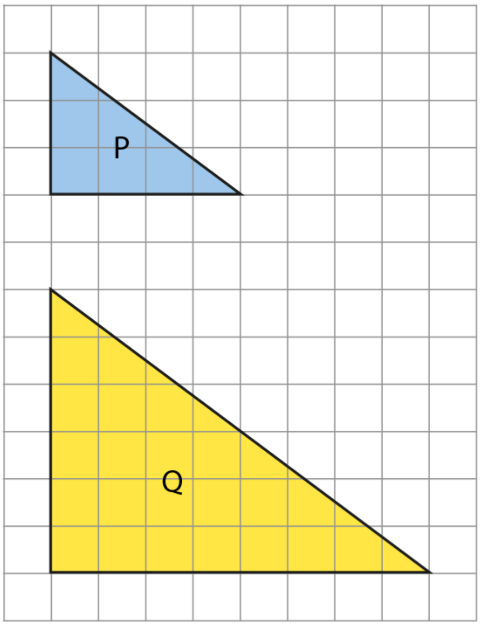
long dimension of D = long dimension of C × **4**

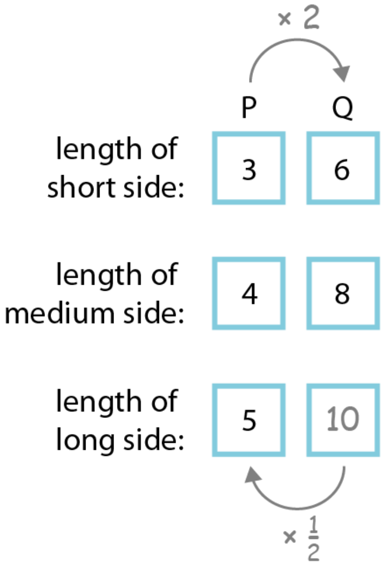
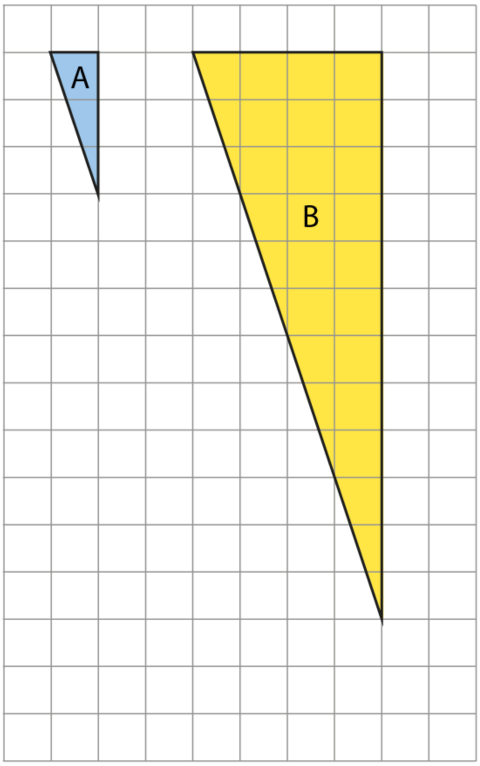
		- Comparing the rectangles: - *'The rectangles are similar because both side-lengths have been scaled by the same scale factor.'* - *'To change rectangle C into rectangle D, scale the dimensions by a scale factor of* ***four****.'* dimensions of C : dimensions of D = 1 : 4 - *'To change rectangle D into rectangle C, scale the dimensions by a scale factor of* ***one-quarter****.'* dimensions of D : dimensions of C = 4 : 1
4:8	Now extend to other irregular polygons, such as right-angled triangles. Use a smaller triangle with dimensions of 3-4-5 units, so that children can easily measure and compare the hypotenuse of the two triangles and see that it has changed by the same scale factor as the other sides.	- *'What scale factor should be used to change the dimensions of triangle P into the dimensions of triangle Q?'* - *'What scale factor should be used to change the dimensions of triangle Q into the dimensions of triangle P?'* 
4:9	Finally, explore how irregular-polygon scaling problems can be solved using the ratio-grid method from *Teaching points 1* and *3*. Begin by revisiting the shapes used in steps *4:7* and *4:8*, as shown opposite, and then solve problems for some other shapes until children are confident with the method.	Example 1 – irregular rectangles: 

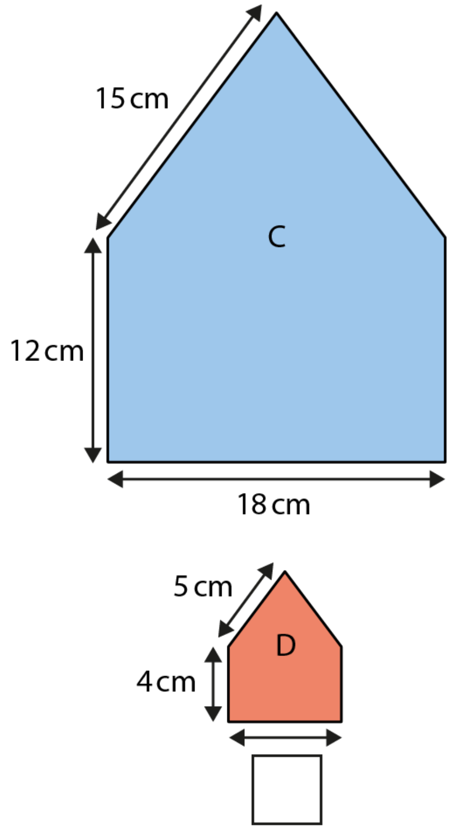


Example 2 – right-angled triangles:

- *'What scale factor should be used to change the dimensions of triangle P into the dimensions of triangle Q?'*
- *'What scale factor should be used to change the dimensions of triangle Q into the dimensions of triangle P?'*
- *'The longest side of triangle P is 5 units. What is the length of the longest side of triangle Q?'*



		
4:10	To complete this teaching point, provide children with some practice-problems involving scaling the dimensions of irregular polygons, such as the examples shown opposite.	• *'What scale factor has been used to change the dimensions of triangle A into the dimensions of triangle B?'* • *'Draw a new triangle, labelled "C", where:'* dimensions of A : dimensions of C = 1 : 3 

		- *'Find the value of the missing length in shape D.'*  *Not to scale.* Dòng nǎo jīn: *'Nita draws a rectangle that has an area of 24 cm^2. She scales the dimensions by a scale factor of three to make a new shape. What is the area of her new shape?'*