Secondary Magazine 128

National Centre for Excellence in the Teaching of Mathematics

Welcome to Issue 128 of the Secondary and FE Magazine

Since no doubt you're surrounded by a frenzy of tinsel and Slade and mince pie fragments, we've tried to keep this month's magazine as soothing as a scented candle twinkling in the frosty calm of a silent night – so no flashing gifs or unexpected Rudolph-related pop-ups, just some ideas about tables (multiplication and binomial probability, not place settings for 12 including those relatives you don't like but feel you have to invite), solving problems (mathematical ones, not long-festering family tensions that always emerge after an eggnog or three), and visual representations of LCMs (that, being cuboidal, may inspire your present-wrapping). Festive cheer, and all that – see you next year.

Contents

Image credit

Heads Up

Whilst not wanting to define this article as mathematical gossip, it does come close! We've brought together news and current mathematical affairs, all in one place. We do hope it will interest you.

Happy Birthday to the theory and equations of General Relativity.

Thank you for all your Christma(th)s books suggestions. Especially popular were

- Patterns of the Universe: A Coloring Adventure in Math and Beauty (Alex Bellos)
- How Not to be Wrong: The Hidden Maths of Everyday Life (Jordan Ellenberg)
- Mathematical Mindsets (Jo Boaler)

and two "We'd never heard of this but we're glad we have now" recommendations were

- Mathematics and Art: A Cultural History Hardcover (Lynn Gamwell)
- Geometric Algebra for Physicists (Chris Doran & Anthony Lasenby).

And what are we at NCETM Central hoping to find under the tree? A pristine copy of Children's Understanding of Mathematics 11-16 (Kath Hart, Dietmar Küchemann et al.) – a fascinating account and analysis of what Hart and her co-researchers discovered in their seminal 1970s research. The conclusions are still pertinent today.

Schools involved in the secondary England-China research project, as part of the Maths Hubs programme, are now starting to try out changes to the way maths is taught, building on the experience of hosting teachers from Shanghai during November. Watch the relevant pages on the Maths Hubs programme website during next term for full reflections of this work, but here and here are a couple of early snapshot views from secondary teachers.

Here at the NCETM, we are constantly impressed by how much maths teachers use the Internet to freely share ideas and resources. One such blog that has caught our eye attempts to pull together all such online offerings linked to teaching the new GCSE. It's been put together by Jo Morgan (aka @mathsjem) on her Resourceaholic website. You might find something you've posted there!

Did you see a headline in the papers last month that suggested it'd take ten years to develop good maths teachers? It came from a report on teacher training by the Advisory Committee on Mathematics Education (ACME). Do you think it was a fair headline?

The DfE is seeking the views of the maths education community in two areas this month. The first concerns the rules and guidance for new AS and A levels in mathematics and further mathematics (first teaching September 2017). And in the second, views are sought on the rules for using calculators in GCSE, AS and A level exams.

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Secondary Magazine 128

National Centre for Excellence in the Teaching of Mathematics

Building Bridges

There's been a lot of debate recently about whether children should memorise the times tables (for example, here, here, here, here and here!). Irrespective of how primary teachers can best ensure that their pupils do know their tables (and some of the most strident objectors are clearly confusing learning and testing), it is undeniable that older pupils in KS3 and 4 are often severely hindered by their lack of knowledge of the multiplicative facts encapsulated in the times tables. Many teachers and teaching assistants with experience of working with pupils with low prior attainment will be reading this, and will know deep down that the root of so many of the barriers to these pupils' progress in their class is a lack of fluency in times tables. It's hard to think of a better example of a mathematical bridge between KS2 and 3 than times tables – and one that in far too many pupils makes the Tacoma Narrows bridge look like the epitome of sturdy stability in comparison!

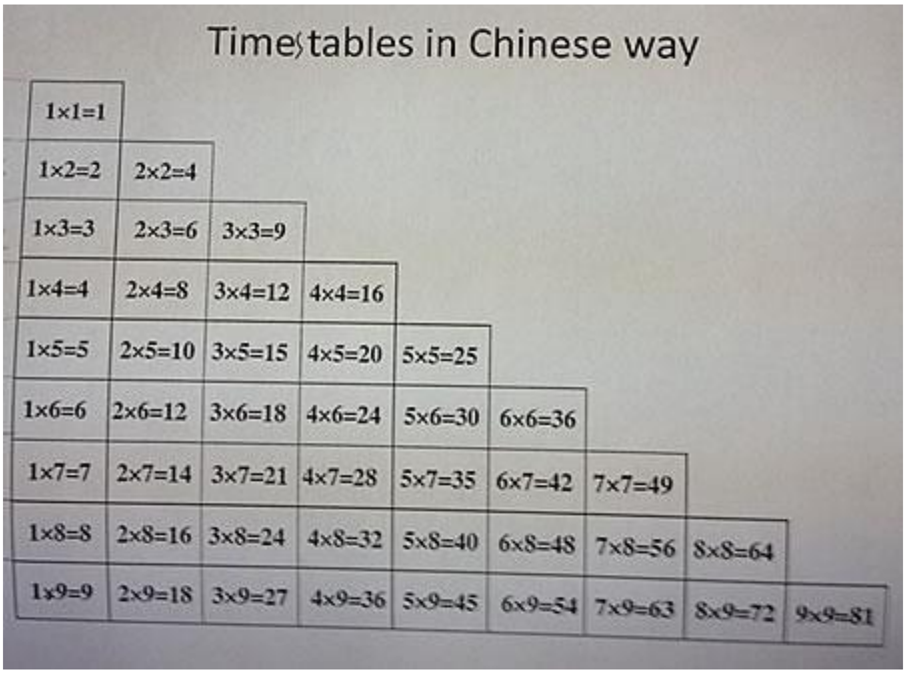
Let's not pretend that if only every pupil in KS3 knew that $7 \times 8 = 56$ then all would be well and no pupil would achieve less than a C grade. John Holt, in *How Children Fail* (an excellent book despite the title!), writes that *"[p]ieces of information like $7 \times 8 = 56$ are not isolated facts. They are parts of the landscape, the territory of numbers, and that person knows them best who sees most clearly how they fit into the landscape and all the other parts of it"*. Importantly, he stresses that knowing times tables is necessary but not sufficient:

> *"The child who has learned to say like a parrot, "7 times 8 is 56" knows nothing of its relation either to the real world or to the world of numbers. He has nothing but blind memory to help him. When memory fails, he is perfectly capable of saying that $7 \times 5 = 23$, or that 7×8 is smaller than 7×5, or larger than 7×10. Even when he knows 7×8, he may not know 8×7, he may say it is something quite different. And when he remembers 7×8, he cannot use it. Given a rectangle of 7cm $\times$ 8cm and asked how many 1cm^2 pieces he would need to cover it, he will over and over again cover the rectangle with square pieces and laboriously count them up, never seeing any connection between his answer and the multiplication tables that he has memorize."*

Nonetheless, factual recall is a start. How liberating would it be if maths topics could be explored without the voyage of discovery getting stuck at base camp needing emergency rations of knowledge of and confidence in times tables? The list of topics that require confident, fluent times tables knowledge is very long indeed, and encompasses a very large proportion of the GCSE syllabus: everyday arithmetic (especially, division), operations with fractions and decimals, using and understanding percentages, multiples, factors, LCM, HCF, algebraic expansion, algebraic factorization, and so on. William Emeny's beautiful network model of the GCSE curriculum makes it clear how important multiplicative confidence is.

So why not set aside the time to hit the times tables where they hurt, and ensure your pupils smash through "the wall"? You might need more time than you expect, and you'll probably want to adopt a "little and often" approach. It seems like a daunting task doesn't it? But when broken down sufficiently, it will be an excellent investment of class time.

What tables facts do they need to KNOW, at their fingertips? Only forty five:



Timestables in Chinese way

1x1=1								
1x2=2	2x2=4							
1x3=3	2x3=6	3x3=9						
1x4=4	2x4=8	3x4=12	4x4=16					
1x5=5	2x5=10	3x5=15	4x5=20	5x5=25				
1x6=6	2x6=12	3x6=18	4x6=24	5x6=30	6x6=36			
1x7=7	2x7=14	3x7=21	4x7=28	5x7=35	6x7=42	7x7=49		
1x8=8	2x8=16	3x8=24	4x8=32	5x8=40	6x8=48	7x8=56	8x8=64	
1x9=9	2x9=18	3x9=27	4x9=36	5x9=45	6x9=54	7x9=63	8x9=72	9x9=81

(see Kangaroo Maths and Trinity Maths for aesthetically more pleasing representations!). This depends on the pupils understanding commutativity, but that's a fundamental concept which they need to grasp irrespective of learning their times tables.

To begin with, let's assume that the pupils in your class know some of the tables in order to help advance to the more difficult ones. For example, if 8 × 7 is required, then perhaps they can recall 8 × 5 and add on two more 8s until reaching 56. However, this writer has observed Year 11 pupils who cannot count on fluently in 8s, let alone recall any of the 8 × table. Where does that leave them? Very stuck indeed! It would take an awful long time to use their fingers to count on one by one.

So let's start at the beginning, with the 2 ×, 10 × and 5 × tables.

- 2 × can be described as doubling, which gives them something familiar to rest on.
- 10 × has the pattern that the digits shift one place to the left and the 'ones' digit becomes a 0. Don't let your pupils say or think that they're adding a zero – that causes havoc when later they face 2.3 × 10.
- 5 × could be approached by linking to 10 ×. Perhaps focus on the even multiples of 5 first, showing how the number can be halved and then the half-number is multiplied by 10 (e.g. to find 6 × 5, note that half of 6 is 3 so 6 × 5 = 30). Then for odd multiples, use the even multiples as a starting point and add 5 (e.g. to find 7 × 5, first work out 6 × 5 as described above, then add 5).

All the time, to develop fluency, keep practising

- commutativity (6 × 5 and then 5 × 6)
- related division facts (if we know that 6 × 5 is 30, then we also know 30 ÷ 6 = 5 and 30 ÷ 5 = 6)
- "empty box" questions with procedural variation (e.g. 7 × ◯ = 42 and ◯ × 9 = 45 as well as 4 × 7 = ◯)

Evidence suggests that an effective sequence for teaching the tables is: 10 ×, 5 ×, 2 ×, 4 ×, 8 ×, 3 ×, 6 ×, 9 ×, 7 ×. So now the 4 × table: a quick way is the 'double double' rule (and divide by 4 can be reviewed at this point using the 'half half' rule), and similarly 'double double double' for the 8 × table. If pupils know some of the 8 × table and want to count on, help them to see that they can add 10 and subtract 2. This is a useful exercise for improving addition skills, and should give them the confidence to count on quickly and accurately.

Onwards!

- Point out that 3 × can be thought of as 'double then add again': 3 × 7 = double 7 and another 7.
- There is a pattern with 6 × which is well worth pointing out because it's a check of accuracy: when multiplying 6 by an even number, the 'ones' digit of the answer matches the multiplying factor e.g. 4 × 6 is 24 and 8 × 6 is 48 (Note that this sounds more rhythmical sounded out as 6 × 4 is 24 and 6 × 8 is 48). For the odd multiples, pupils can work from the even multiples and add 6 (or, add 5 and then 1).
- 9 × is popularly calculated using the finger method (yes, you know the one – loved by pupils, hated – unreasonably, it's great for fluency – by many teachers). Counting on 9s should be done by counting on 10s and subtracting 1 each time; this is good practice for adding "near 10s" (e.g. adding 9, 19, 29 etc. to a number): well worth a review.

And finally…

- 7 ×. But there's only 7 × 7 left! So why not do a review of square numbers at the same time as 7 × 7 to reinforce 12 to 102?

After reinforcing each times table individually, overall fluency in recall will only be achieved by mixing up questions randomly and gradually building in more and more individual times tables to the mix: try the spreadsheets 60 Club and 100 Club.

Why not get a competition going to inspire more and more of your pupils to join the club!? Fluency in the times tables could be judged as 100 correct (for the 100 Club) in 2½ minutes or 60 correct (for the 60 Club) in 1½ minutes – but reserve the Gold medals for the pupils with "division fluency" (56 ÷ 7 = ◯ and 63 ÷ ◯ = 9 and ◯ ÷ 4 = 7) as well as "multiplication fluency" (8 × 9 = ◯ and 6 × ◯ = 30).

There is a myriad of websites out there to focus on times tables, both free and subscription, for example:

- Woodlands Junior School's Interactive Times Tables Games;
- to display patterns within the times tables on the board use this from Maths-Resources.com - great for identifying the symmetry within the times tables, square numbers and just how few times tables may be causing issues;
- if your school is willing to spend a small amount of money on a subscription, look no further than Times Tables Rock Stars, which really gets the enthusiasm going through avatars, games and competitions.

Let us know what you try and what you like: email info@ncetm.org.uk or share a picture on Twitter @NCETMsecondary.

You can find previous *Building Bridges* features here.

Image credits

Sixth Sense

Last month, we sketched – almost literally – a possible route through all of the material with which students need to be fluent if they are to understand conceptually the Binomial Distribution; this month we're going to look at Binomial Hypothesis Testing. For some students of the current specifications, this idea is not covered until the S2 module, but the new specifications will include this in AS Mathematics so all students of the subject, even those not intending to sit an A Level, will be expected to carry out and interpret hypothesis tests.

From the statistician's point of view, a hypothesis test has five stages:

1. propose hypotheses
2. set significance level
3. calculate critical region
4. carry out experiment
5. conclude

With a carefully-designed initial experiment (practical, ideally, but thought-experiment if necessary) it should be possible to elicit the key ideas from your students without giving them this list.
For example, you could tell your students that you are going to give them a die, which you may or may not have tampered with. If you *have* tampered with it, you have done so in such a way as to make "5" come up less frequently than they would think it should. How might they find out whether or not you have tampered with it? Alternatively, you could propose implied hypotheses which require the collection of data from, or by, the students. For example, "birth rates are lower in the first 8 months of the year" could lead to hypotheses testing the value of p where p = P(a person is born between 1 January and 31 August). Whatever you decide, it is important to try to start with a situation that will require a "one-tailed test" at the lower end – we don't want to make life too complicated right from the start!

Let's explore the first example in a bit more detail. The conversation never goes quite like this, but with a bit of steering by you, the key ideas can be elicited from your students!

Teacher: I'm going to give you a hypothetical die.

Student 1: What's a die?

Teacher: I'm being pedantic – we often call it a dice, but that's technically the plural. Anyway, either it "behaves fairly" or I've tampered with it so that "5" comes up less frequently than I think you think it should. How will you find out?

Student 2: Roll it lots of times and count the 5's.

Teacher: Good: how many times?

Student 3: If we roll it sixty times, we'd expect "5" to be rolled ten times.

Teacher: What do you mean by that? Will we get exactly ten 5's?

Student 3: No. Well, we might. But I mean that rolling ten 5's is the most likely result, but that there are plenty of other things that could happen.

Student 4: So if we roll it sixty times and get "5" nine times, will we say "you have tampered with this dice?" I don't think we should, because this could actually happen. Then again, anything *could* happen, but if we're way off getting ten 5's then we're going to be suspicious!

Teacher: Interesting – so at what point are you prepared to stand by an accusation of tampering? What's "way off?"

Student 5: I'd say if we get "5" six times.

Student 6: And if we get "5" fewer times than that, surely?

Teacher: OK, so we're saying that our decision process is:

1. Roll the die sixty times.
2. Count the number of 5's.
3. And if we get six or fewer 5's, then you'll accuse me of die-tampering.

Let's think about this happening. If you roll a fair die sixty times, what is the probability that we roll "5" six times or fewer?

Student 7: Let me consult my tables. Oh no, they only go up to n = 20.

Teacher: Fine, let's build a 60-table on a spreadsheet. [This doesn't take long and is an easy formula for students to see. Use two columns: one contains the integers from 0 (in cell A1) to 60 (in cell A61), then type "=binom.dist(A1,60,1/6,1)" into cell B1 and drag this down column B]

Student 8: Under the modelling assumption that "no. of 5's rolled" $\sim B\left(60,\frac{1}{6}\right)$ we can see from the spreadsheet that P(no. of 5's rolled ≤ 6) = 0.1081 (4d.p.)

Teacher: OK. So when we roll the die and get only four 5's, you're going to accuse me of die-tampering. Will you be certain that I've done so?

Student 9: No, because we might just be unlucky. Fair coins do come up "Heads" 5 times in a row, it's not impossible. Fair dice can be rolled and not give the scores you're expecting. Even if we get no 5's at all that's not PROOF that you've tampered with the die.

Teacher: Correct, so this means that if you accuse me of tampering, you know you might be making a false accusation – in fact the probability of you making a false accusation is…

Student 10: …0.1081, because I'll falsely accuse you if it is a FAIR dice, modelled with P(roll a 5) = 1/6, and genuinely I roll "5" six or fewer times out of sixty, and the probability of that is what we just worked out .

Student 11: But that's way too high for the risk of a FALSE accusation like this – I'm not comfortable with that. I'd rather risk a false accusation with a probability closer to 1% than 10%.

Student 12:	In which case, looking at our table, we'd have to make an accusation if "three or fewer" 5's are rolled – according to the table, the probability of this happening is 0.0063 which is under 1%, but if we decided to accuse if "four or fewer" the probability is nearly 2%.
Teacher:	Great, lots of wriggle room: you've just made it easier for me to get away with giving you a tampered die! So clearly there's a bit of a balancing act here between "making a false accusation too frequently" and "making it too easy for someone to get away with tampering". Clear the table, let's start rolling!
INTERLUDE	
Student 13:	So we rolled the die sixty times, and rolled a 5 (drum roll please) five times. We said we'd accuse you of tampering if we got three or fewer 5's, which we haven't, so we won't accuse you.
Teacher:	Hooray, I'm a free man. But do you KNOW that I haven't tampered with the die?
Student 14:	No. Maybe you did, and were lucky and got away with it, like a "false negative" when you have a test for a nasty disease.
Teacher:	If we'd only rolled a 5 twice, would you KNOW that I had tampered with the die?
Student 15:	No. Maybe you hadn't, but you were unlucky, like a "false positive" in a medical test.

At this point, your students are ready to use some notation and terminology, and to formalise what they have learned so far.

1. A statistical "hypothesis test" starts with two hypotheses. In our example, they are

   The null hypothesis – H_0: $p = \frac{1}{6}$

   The alternative hypothesis - H_1: $p < \frac{1}{6}$

   where p represents what we think is the probability of rolling a 5.

2. Let's decide that in this instance, we accept the risk of making a false accusation (of the type "you've tampered with the die when actually (unknown to us) you haven't") with probability 0.1. We call this the *significance level*. So, here we're setting a "10% significance level". This seems acceptable (but not to everyone in the dialogue above) but clearly this will vary from experiment to experiment. Statisticians have to make the decision based on context; students are likely to be told what to use in the exam question.

3. The *critical region* is the set of outcomes that result in our rejecting H_0: that is, making an accusation (which may or may not be a correct accusation). It is, therefore, the set of outcomes that lead us to make a FALSE accusation if H_0 is, in fact and unknown to us, true.

   So, in our experiment, assuming that "no. of 5's rolled" $\sim B\left(60, \frac{1}{6}\right)$, we can see from our table (as discussed earlier) that "P(no. of 5's rolled $\leq$ 6) = 0.1081"

This is a potential sticking point, but a good example – having picked a significance level of 10%, this figure is actually too high. We could change our level to 11%, but we ought not to be influenced by the figures involved. Having set the level to 10%, we revisit the table to see that "P(no. of 5's rolled ≤ 5) = 0.0512". This means that our proposed significance level of 10% is actually a significance level of 5.12% in this experiment, because the probabilities accumulate "chunkily".

Thus our critical region is "no. of 5's rolled is 5 or fewer", which we can write in set notation as $\{X \leq 5\}$, defining X to be (the random variable) the number of 5's rolled when the die is rolled sixty times.

4. Now, and only now, are we ready to "do the experiment". All this thinking should have happened so as not to influence our decision at the end. If we do the experiment first, we could be tempted to tweak our critical region to fit the experimental outcome.

   So we roll the die sixty times and get…

5. a) …7 "5's" and 53 "not 5's".

   Conclusion: 7 IS NOT in the critical region. There is insufficient evidence, at the 10% significance level, to suggest that the die has been tampered with. Therefore, we are not comfortable making an accusation. We say that "we accept H_0". This doesn't mean we are sure that the die is fair, but we haven't got enough evidence to be comfortable suggesting otherwise.

   b) …5 "5's" and 55 "not 5's".

   Conclusion: 5 IS in the critical region. There is sufficient evidence, at the 10% significance level, to suggest that the die has been tampered with. Therefore, we are comfortable making this accusation. We say that "we reject H_0."

Students should now be ready to tackle some questions. It is a good idea to set, at least initially, questions that follow the above structure – exam questions will necessarily contain the experimental result but we want students to set up the test **first**, **before** being influenced by this. Consequently, a bit of editing may be useful, so that questions look something like:

It is believed that three-quarters of people living in Manchester use the trams at least once a week. A researcher wishes to find out whether or not a campaign to encourage more people to walk, cycle and use the bus has reduced the proportion of Manchester-inhabitants using the trams. As such, she decides to carry out a survey of 20 such people.

(a) What are her hypotheses?
(b) If she chooses a 5% significance level, what is her critical region?

12 of the 20 people surveyed say they use the tram at least once a week.

(c) What should she conclude?

One of them contacts the researcher to say that he was mistaken, and he should have said "no, I do not use the tram at least once a week".

(d) What should the statistician conclude now?

Only once students have grasped the procedural fluency of the “five step” structure is it worth altering the questions so that they have to cope with the irritating exam question format that gives the experimental result in the first sentence!

Further lessons on upper tail tests, and two-tail tests are, of course, necessary but it isn’t worth rushing into these. Even students who are happy with the process, and who have previously mastered the art of calculating all sorts of probabilities from the tables, find the “upper tail” argument difficult:

For hypotheses H_0: $p = \frac{2}{5}$

H_1: $p > \frac{2}{5}$

and an experiment involving 16 trials, we find ourselves looking at the following table:

	0.4
8	0.857730282
9	0.941681055
10	0.980858082
11	0.995104274
12	0.99906155
13	0.999873298
14	0.999989263
15	0.999999571
16	1

Clearly $P(X \le 9) = 0.9417$ so we have to consider the “opposite” of this – we deduce that, because $P(X \le 9) < 95\%$ it must be the case that $P(X > 9) > 5\%$ which is too big given our significance level.

Instead, students must argue that $P(X \le 10) > 95\%$

so $P(X > 10) < 5\%$

and thus the critical region here is X, the thing being counted, needs to happen 11 or more times in 16 trials – i.e. $\{X \ge 11\}$ is critical region.

Not straightforward, and certainly I would teach and embed lower tails, then have a few lessons on something else, and then come back to upper and two-tails later!

You can find previous *Sixth Sense* features here.

Image credit

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From the Library

"I've never seen a question on pyramids like that before. We've not been taught that," said one of my star pupils as she left a mock exam. Did I detect in her eyes the look of one who had been betrayed? A pang of guilt struck. But then, I can't teach them every possible problem that might come up – but am I supposed to? Do my pupils *assume* I do, and think that there'll be no surprises, no "now what do I do now?" moments, in the exam?

How to teach the unexpected? Or how to teach *how to handle* the unexpected. Problem solving, with its 30% element in the reformed GCSE, tackles the issue of dealing with "complex and unfamiliar problems" (Watson 2014) where "the methods to use are not obvious or there may be a choice of different methods" (Schoenfeld 1992). But for pupils to learn how to problem solve requires them to do the exploring, the thinking, the discovery. A transmission mode of teaching, where learning "is an individual activity based on watching, listening and imitating until fluency is attained" (Swan and Swain 2010) may not provide the experience the pupils need to develop their problem solving skills. The teaching needs to become pupil-centred – and, according to Ofsted (2008, 2012), for many of us this requires a change of teaching style. But teachers' professional development itself "has been described as an 'unsolved problem', particularly where there is an expectation to change teaching practices from teacher-centred orthodoxies to more pupil-centred approaches" (Watson 2014). A couple of research papers attempt to shed some light on what works.

Swan and Swain (2010) implemented a programme "designed to challenge existing practices and beliefs by investigating how teachers might incorporate the following pedagogical principles into their teaching", including:

- exposing common misconceptions
- promoting explanation, application and synthesis rather than mere recall
- encouraging reasoning rather than 'getting answers'
- creating connections between topics, so that pupils do not see mathematics as a set of unrelated tricks and techniques to be memorised

all in the context of collaborative work with an emphasis on discussion. Crucially, rather than informing teachers of the theories they would need to embrace, instead

- their existing beliefs and practices were recognised
- contrasting practices viewed and conflicts discussed
- different approaches experimented with and supported
- time for self-reflection and discussion provided.

The researchers witnessed practices becoming less transmission-oriented, supporting their view that "changes in beliefs are more likely to follow changes in practice, after the implementation of well-engineered, innovative methods, as processes and outcomes are discussed and reflected upon."

Watson (2014) viewed professional development through the idea that learning is dependent on three key factors: (1) direct observation of the desired behaviour, (2) an individual's belief in their potential success, and (3) the coherence between an individual's beliefs, the social context and the individual's behaviour. Watson highlights the importance of appropriate models, suggested-lesson plans, video examples, and suitable classroom activities, as well as the individual's self-belief, in implementing the suggested approach. He suggests that moving from a teacher-centred approach is constrained by the demands of day-to-day teaching as well as the attraction that it provides “routines that students, parents and teachers have familiarity with”.

There are many courses and resources (see below) now available for developing the skills for facilitating problem solving. The ideas in these research papers may help make this a reality in the classroom. And perhaps we can also make our pupils happier: "I can't remember the numbers, but the one about Hannah's sweets in particular made me want to cry." None of us wants this to be our pupils’ abiding memory of GCSE maths.

What helps a teacher create a class full of confident problem solvers? How have you managed to achieve this? Let us know, by email to info@ncetm.org.uk or Twitter @NCETMsecondary.

Resources

Many resources are available for Professional Development, a selection is listed here:

- NCETM Departmental Workshops
- Bowland Maths
- PRIMAS, a European project to promote inquiry-based learning in mathematics and science
- FRESH courses from MEI
- NRICH - A Guide to Problem Solving
- For another angle on “problem-solving” see the Inquiry maths website.

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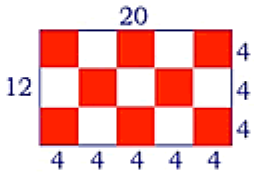
You can find previous *From the Library* features here.

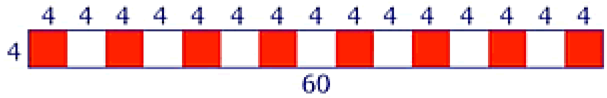
Image credit

It Stands to Reason

In Issue 127 we explored certain visual images that can help pupils develop conceptual understanding about factors and multiples, and hence reason about them with confidence and deeper perception. The easily observed structure of *images* reveals the much less obvious *numerical* structure that pupils can then exploit. Our aim is to help our pupils acquire the depth of understanding that enables them to use key relationships flexibly, so that they are not restricted to following without thinking algorithmic routines.

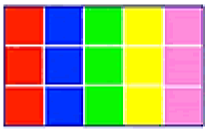
So let us continue from where we left off! Look at these two diagrams: the 15 square tiles of side-length 4 that fit in the 12-by-20 rectangle on the left have been rearranged to form the 4-by-60 rectangle on the right …

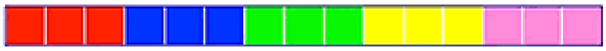


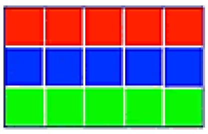


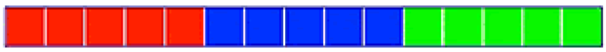
Having reasoned (see last issue) from the 12-by-20 rectangle, that 4 (which is now represented by the height of the 4-by-60 rectangle) is the highest common factor of 12 and 20 [HCF(12, 20)], pupils should then be asking themselves: what does the ***width*** of the 4-by-60 rectangle represent?

Give pupils time to think and discuss their thoughts about the answer to this question. The aim is for them to connect the ***width*** of the rectangle (as well as its ***height***) to the numbers 12 and 20. They might find it helpful to make sketches, and then colour them in significant ways, such as these …









The key is seeing that …

$60 = (3 \times 4) \times 5 = \mathbf{12} \times 5$
$60 = 3 \times (4 \times 5) = 3 \times \mathbf{20}$

… from which it follows (3 and 5 having no common factors) that the ***width*** of the 4-by-60 rectangle is the least common multiple of 12 and 20 [LCM(12, 20)]. If pupils know a definition of the least common multiple of two numbers *p* and *q*…

> *The least common multiple of p and q, denoted LCM(p, q), is the smallest positive number, m, for which there exist positive integers n_p and n_q such that $p \times n_p = q \times n_q = m$.*
> [Wolfram Mathworld]

… they should be able to see, and articulate reasoning to explain, exactly why the ***width*** of a rectangle with area $p \times q$ and height equal to HCF(p, q) MUST BE the least common multiple of p and q. This is how they might reason about this example:

- The area of a 12-by-20 rectangle is (3 × 4) × (5 × 4).
- The smallest possible number which has factors 3 × 4 and 5 × 4 is 3 × 4 × 5, and 3 and 5 have no common factors. So LCM(12, 20) is 3 × 4 × 5.
- We also know that 4 is HCF(12, 20).
- If we draw a rectangle with the same area as a 12-by-20 rectangle, that is with area (3 × 4) × (5 × 4) = (3 × 4 × 5) × 4, but with height 4 (which is HCF(12, 20)), the width must be 3 × 4 × 5, which is LCM(12, 20).

One powerful consequence of this representation is that pupils can use relatively simple facts that they know about shapes to deduce a more complicated fact, one they probably did not previously know, about *numbers*:

- Since both rectangles are formed with fifteen 4-by-4 squares they have the same area.
- Therefore the area of the 4-by-60 rectangle is 12 × 20.
- The area of a rectangle is its height × its width.
- The height of the 4-by-60 rectangle is HCF(12, 20).
- The width of the 4-by-60 rectangle is LCM(12, 20).
- Therefore HCF(12, 20) × LCM(12, 20) = 12 × 20.

Thus they have deduced an example of the general relationship that

HCF(p, q) × LCM(p, q) = p × q.

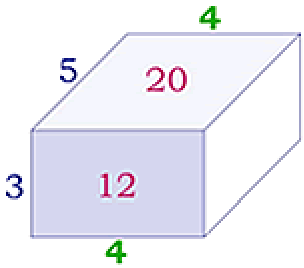
This is the general reasoning (which pupils may not yet be ready to develop unsupported, but should be ready to explore with guidance):

Suppose HCF(p, q) is h.

- Then p = a × h and q = b × h, where a and b have no common factors (otherwise h would not be the highest common factor of p and q.)
- The area of a p-by-q rectangle is (a × h) × (b × h).
- The smallest possible number which has factors a × h and b × h is a × b × h (remember a and b have no common factors). So a × b × h is LCM(p, q).
- If we draw a rectangle with the same area as a p-by-q rectangle, that is with area (a × h) × (b × h) = (a × b × h) × h but with height h which is HCF(p, q)., the width must be a × b × h, which is LCM(p, q).

Cuboids

Quite different spatial reasoning about a 3-by-4-by-5 cuboid leads to, and so further illuminates, the same conclusion.



- The numbers 12 and 20 are represented by the areas of pairs of opposite faces of the cuboid.
- The HCF of 12 and 20, which is 4, is represented by one edge-length of the cuboid.
- The 'other' factors of 12 and 20 (which are 3 and 5 respectively) are represented by the other two edge-lengths of the cuboid.
- The volume of the cuboid, which is 5 × the area of the 3-by-4 face or 3 × the area of the 5-by-4 face then represents the LCM of 12 and 20, which is 3 × 4 × 5 = 60.

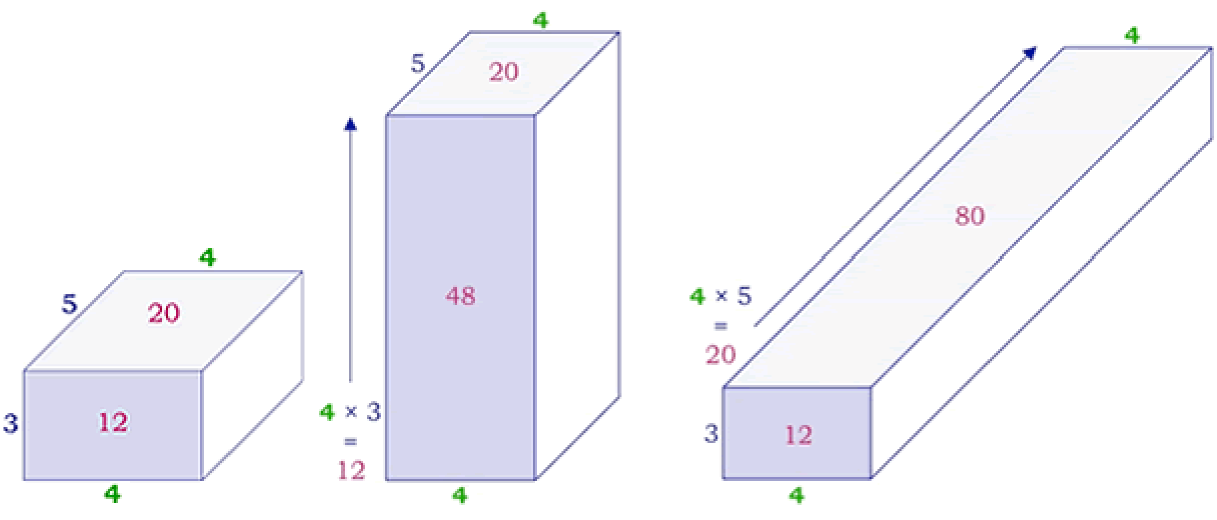
As in the previous example using rectangles, pupils have an opportunity to construct a chain of reasoning that shows why this finding is a ***necessary consequence*** of the given assumptions:

- they know that 12 = 3 × 4 and 20 = 5 × 4, and that 4 is HCF(12, 20). By multiplying 3 × 4 by 5, or by multiplying 5 × 4 by 3, they must make the least common multiple of 3 × 4 and 5 × 4 because 3 and 5 have no common factors. So LCM(3 × 4, 5 × 4) = 3 × 4 × 5.
- the volume of a 3-by-4-by-5 cuboid is 3 × 4 × 5 (and by-the-way the areas of two pairs of opposite faces are 3 × 4 and 5 × 4)
- so the volume is LCM(3 × 4, 5 × 4) = LCM(12, 20).

In general:

- they know that, if p = a × h and q = b × h where h is HCF(p, q), then LCM(p, q) is a × b × h
- the volume of a cuboid with edge-lengths a, b and h is a × b × h. So it represents the LCM of p and q.

Pupils should know that in order to multiply the volume of a cuboid by *n*, they must multiply just one dimension of the cuboid by n. Therefore to represent LCM(12, 20) × HCF(12, 20) we must stretch the cuboid by a factor of 4 in one direction only …

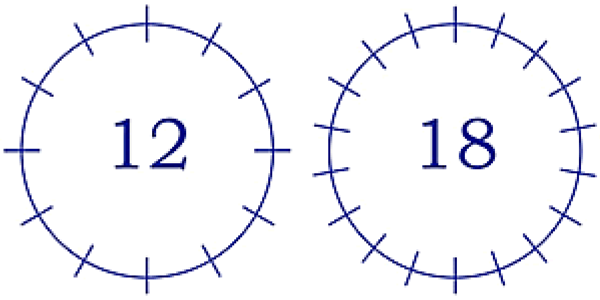


Both of the images of the cuboid-after-it-has-been-stretched-in-one-direction-only show that 4 × the volume of the original cuboid = 12 × 20. So they both show that HCF(12, 20) × LCM(12, 20) = 12 × 20.

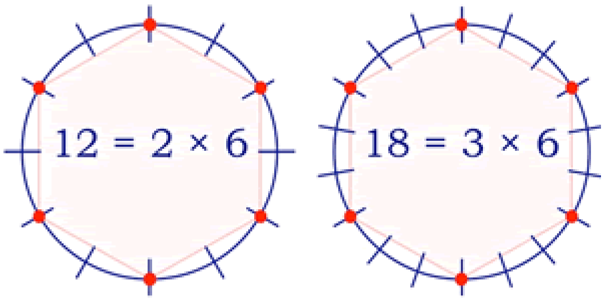
Polygons in circles

Polygons created by joining equally spaced points on the edge of a circle are a different set of visual images that can help pupils see, and reason about, highest common factors and lowest common multiples.

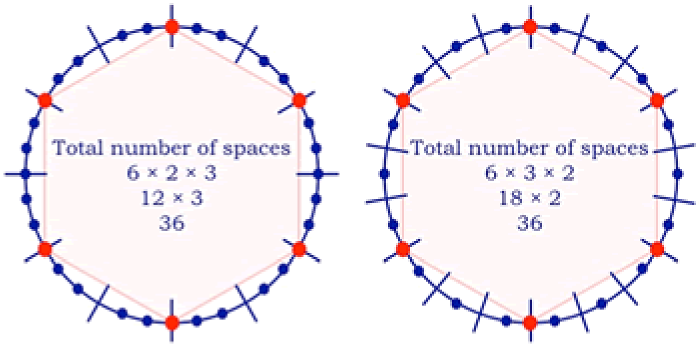
When pupils consider 12 equally spaced points marked on one circle, and 18 on another …



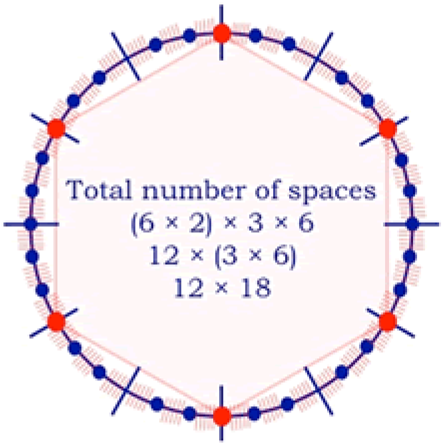
… they can look for the greatest possible number of sides of a regular polygon that can be drawn in both circles with all its corners on marked points. That number of sides must be the greatest factor of both 12 and 18 (which are the number of points on each circle); that is, it must be HCF(12, 18). So, by making sketches of regular polygons in both circles, they will see that HCF(12, 18) = 6:

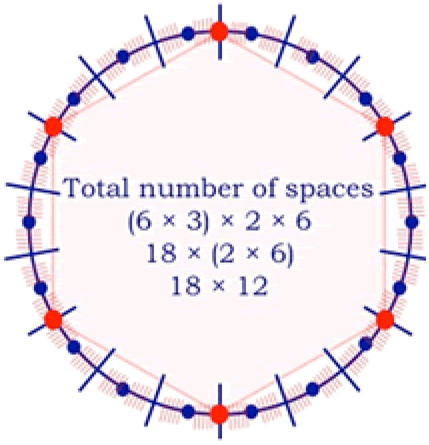


The number of equal spaces on the circle's edge between each joined dot represents in one circle the fact that 12 = 2 × 6, and in the other that 18 = 3 × 6. If on the circle with 2 spaces between joined dots we now split every one of these spaces into 3 equal parts, and on the circle with 3 spaces between joined dots we now split every one of these spaces into 2 equal parts, we create the same total number of spaces on both circles. What does that number of spaces represent?



If we now split **every** space on **both** circles into 6 (the HCF of 12 and 18) equal parts [as we have done in the following diagrams – although the individual marks are difficult to see!], the total number of spaces is the product of 12 and 18.





These images illustrate the fact that HCF(12, 18) × LCM(12, 18) = 12 × 18.

Having reached the same conclusion about how the HCF, the LCM and the product of two particular numbers are related by reasoning about features of three completely different geometric images, your pupils should be convinced that it is true for those particular numbers. In the first two image-contexts pupils used simple geometrical facts that they knew to support them in reasoning as follows:

- 12 = 3 × 4 and 20 = 5 × 4
- HCF(3 × 4, 5 × 4) [which is HCF(12, 20)] = 4
- 3 and 5 have no common factors
- LCM(3 × 4, 5 × 4) [which is LCM(12, 20)] = 3 × 5 × 4
- So HCF(12, 20) × LCM(12, 20) = 4 × (3 × 5 × 4) = (4 × 3) × (5 × 4) = 12 × 20

The third set of images helped them to reason that:

- 12 = 6 × 2 and 18 = 6 × 3
- HCF(2 × 6, 3 × 2) [which is HCF(12, 18)] = 6
- 2 and 3 have no common factors
- LCM(2 × 6, 3 × 6) [which is LCM(12, 18)] = 2 × 3 × 6
- So HCF(12, 18) × LCM(12, 18) = 6 × (2 × 3 × 6) = (6 × 2) × (3 × 6) = 12 × 18

By focusing on, and discussing, the structure that is common to the numerical reasoning stimulated by visual images such as we have discussed, pupils can be supported to understand, and possibly construct using algebraic notation, the general argument below. They will see that in order to justify the general conjecture …

HCF(p, q) × LCM(p, q) = p × q

… they have always reasoned as follows:

- Suppose p = a × h and q = b × h, where h is HCF(p, q) and a and b have no common factors

- Then LCM(p, q) = a × b × h
- HCF(p, q) = h, so HCF(p, q) × LCM(p, q) = h × a × b × h
  = (a × h) × (b × h)
  = p × q.

Your pupils are more likely to suggest, and follow, and recall, this general **abstract** argument after they have had sufficient opportunities to reason about **concrete and pictorial** representations of the HCF, LCM and product of particular values of p and q in particular visual images. In using such images to reason about pairs of numbers such as 12 and 20, pupils have first ***specialised*** in order to develop a clearer sense of the ***general*** argument.

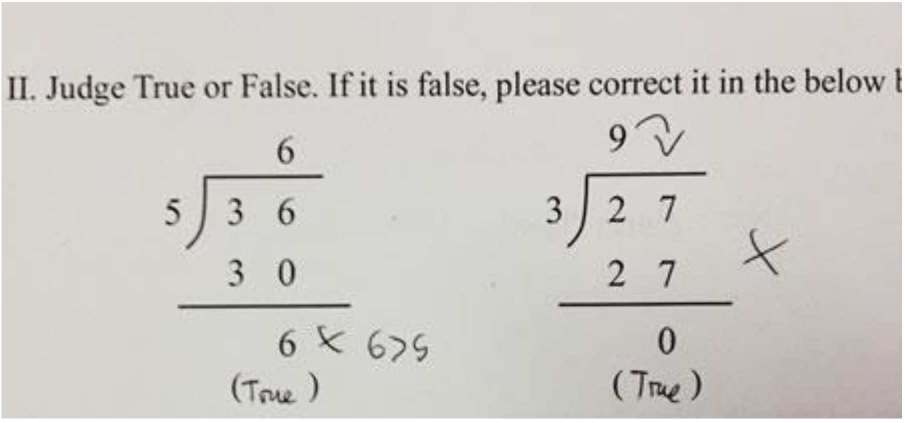
Not only is the general relationship between the highest common factor, least common multiple and product of two numbers a conclusion that pupils can be challenged to reach through reasoning – and thereby deepen their understanding – but, once established and therefore known by them, it is a fact that they can and should use: their conceptual understanding should lead them to **procedural fluency**. For example, suppose that they want the least common multiple of 27 and 60. If they remember that LCM = product ÷ HCF, they can simply multiply 27 × 60 = 1620 and divide by the HCF, which is easily seen to be 3 (recall that the HCF must be a factor of the non-zero difference between any two multiples of the original numbers, and 60 – 27 × 2 = 6). LCM(27, 60) = 540 then follows.

You can find previous *It Stands to Reason* features here

Image credit

Eyes Down…

Asking pupils to decide if a mathematical statement or argument or chain of reasoning is true or false is a worthwhile and commonly-used activity. This particular example, given to a Year 7 class at the Nobel School in Stevenage, struck me because of the close attention to detail that the pupils had to pay. These divisions were both obviously true (which is what the pupil first decided, at the bottom of the picture), and only after looking again, and more carefully, did doubt set in!



Have you an example of a similar activity (perhaps an *Odd One Out* or an *Always Sometimes Never* question) that is similarly forensic?

If you have a thought-inducing picture, please send a copy (ideally, about 1-2Mb) to us at info@ncetm.org.uk with 'Secondary Magazine Eyes Down' in the email subject line. Include a note of where and when it was taken, and any comments on it you may have. If your picture is published, we'll send you a £20 voucher.

You can find previous *Eyes Down* features here

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