MathsHUBS Resources

National Centre for Excellence in the Teaching of Mathematics

Mastery Professional Development

Multiplication and Division

2.10 Connecting multiplication and division, and the distributive law

Teacher guide | Year 4

Teaching point 1:

Multiplication is commutative; division is not commutative.

Teaching point 2:

Multiplication is distributive: multiplication facts can be derived from related known facts by partitioning one of the factors, and this can be interpreted as partitioning the number of groups; two-part problems that involve addition/subtraction of products with a common factor can be efficiently solved by applying the distributive law.

Teaching point 3:

The distributive law can be used to derive multiplication facts beyond known times tables.

Overview of learning

In this segment children will:

- review, in detail, the fact that multiplication is commutative (the relationship can be expressed as $a \times b = b \times a$) through the interpretations already explored:
  - the 'one interpretation, two equations' application of commutativity; for example, three groups of five can be represented by both $3 \times 5 = 15$ and $5 \times 3 = 15$, and so $3 \times 5 = 5 \times 3$ (see segment *2.3 Times tables: groups of 2 and commutativity (part 1))*
  - the 'one equation, two interpretations' application of commutativity; for example, $3 \times 5 = 15$ can represent three groups of five, or five groups of three (see segment *2.5 Commutativity (part 2), doubling and halving*)
- use the relationship between multiplication and division to conclude that division is *not* commutative, and apply the inversion principle, i.e.:
  if $\boldsymbol{a \times b = c}$, then we can write
  - $\boldsymbol{c \div b = a}$
  - $\boldsymbol{c \div a = b}$
- explore the distributive law of multiplication (i.e. $\boldsymbol{a \times b + a \times c = a \times (b + c)}$) including:
  - building on their understanding of the difference between adjacent multiples to explore how known multiplication facts can be used to derive other products, by adding or subtracting either one group or several groups
  - using the distributive law to solve a variety of abstract and contextual problems
  - using the distributive law to calculate products involving numbers beyond known times-table facts.

Children should already have a secure understanding of both interpretations of multiplicative commutativity, as described above. At the beginning of *Teaching point 1*, learning about commutativity is brought together and reviewed; this serves not only as consolidation of previous learning, but also as preparation for learning about the non-commutativity of division and application of the inversion principle. To support children in making connections, the same context is used both to review multiplicative commutativity and to explore division. Investigation of the context is used to demonstrate how the product in the multiplication equation (e.g. $3 \times 5 = \boldsymbol{15}$) always translates to the dividend in the corresponding division equations; two valid division equations can be written to correspond to the multiplicative context ($\boldsymbol{15} \div 5 = 3$ and $\boldsymbol{15} \div 3 = 5$) – which number becomes the divisor and which becomes the quotient depends on the particular problem being solved.

Children will further investigate the validity of division equations via the inversion principle in the special case where one of the values is equal to zero, generalising that we should never write a calculation where the divisor is zero.

Teaching point 2 covers the distributive law of multiplication in detail. Children have already used the distributive law informally, in the following contexts:

- For each times table learnt, children have noticed and applied the fact that adjacent multiples of a number have a difference equal to that number (e.g. $5 \times 2 = 4 \times 2 + 2$ and $4 \times 2 = 5 \times 2 - 2$).
- For some times tables (six, seven and nine), children have learnt how to use arrays to derive 'new' multiplication facts from known facts (for example, relating 7×7 to 5×7 and 2×7).

When considering the distributive law in a grouping context, it is possible to connect the factor being partitioned with:

- the *number of groups*; for example, five groups of eight is equal to three groups of eight plus two groups of eight

or

- the *group size*; for example, five groups of eight is equal to five groups of five and five groups of three.

Since the former is more accessible to children, and builds directly on their understanding of the relationship between adjacent multiples, this approach is used in *Teaching point 2*, until children are confident working with abstract equations alone without the support of a concrete or pictorial grouping representation.

The focus then shifts to the application of the distributive law. including using it:

- as a strategy to help perform any multiplication calculation where one of the factors is six, seven or nine, irrespective of whether the common factor represents the number of groups, the group size or neither (the latter in the case of non-contextual problems)
- to solve contextual problems where the factor being partitioned represents the group size
- to efficiently solve two-part problems (contextual or abstract) with a common factor; for example solving $9 \times 4 + 9 \times 5$ by calculating 9×9.

Teaching point 3 briefly extends application of the distributive law to derivation of multiplication facts where one of the factors is a teen number (e.g. $7 \times 13 = 7 \times 10 + 7 \times 3$), directly preparing children for segment *2.11 Times tables: 11 and 12*. Once children have learnt how to multiply by a multiple of ten (segment *2.13 Calculation: multiplying and dividing by 10 or 100*), the same strategy can be extended to two-digit numbers to support efficient calculation (e.g. $7 \times 23 = 7 \times 20 + 7 \times 3$).

An explanation of the structure of these materials, with guidance on how teachers can use them, is contained in this NCETM podcast: www.ncetm.org.uk/primarympdpodcast. The main message in the podcast is that the materials are principally for professional development purposes. They demonstrate how understanding of concepts can be built through small coherent steps and the application of mathematical representations. Unlike a textbook scheme they are not designed to be directly lifted and used as teaching materials. The materials can support teachers to develop their subject and pedagogical knowledge and so help to improve mathematics teaching in combination with other high-quality resources, such as textbooks.

Teaching point 1:

Multiplication is commutative; division is not commutative.

Steps in learning

1:1

In this teaching point, we will revisit the commutative law of multiplication to explore the connections between multiplication and division, and to highlight that multiplication of two numbers can be done in any order (multiplication is commutative), whereas division of one number by another cannot.

Briefly recap the commutative property of multiplication (exemplified below) as:

- 'one interpretation, two equations' (see segment *2.3 Times tables: groups of 2 and commutativity (part 1)* , *Teaching point 5*)
- 'one equation, two interpretations' (*see segment 2.5 Commutativity (part 2), doubling and halving, Teaching point 1*).

Also, ensure that children can work with abstract expressions alone (in the absence of a grouping context), by identifying expressions that have the same product. Use the stem sentence:

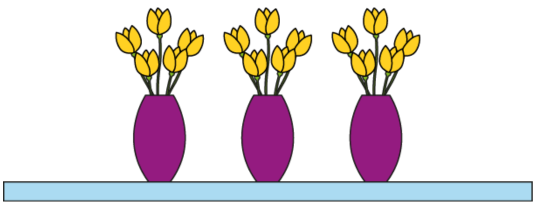
'The product of ___ and ___ is equal to the product of ___ and ___.'

This can then be simplified to: ***'___ times ___ is equal to ___ times ___.'***

Encourage children to explain the relationships by making or drawing arrays.

Commutativity of multiplication – 'one interpretation, two equations':

'Write two multiplication equations to represent this picture.'



$3 \times 5 = 15$ • *'Three groups of five are equal to fifteen'*	$5 \times 3 = 15$ • *'Five, three times is equal to fifteen.'*

- *'Three groups of five can be written as three times five, or as five times three.'*

- *'What's the same?'*
  'In both equations "3" and "5" are factors and "15" is the product.'
- *'What's different?'*
  'The factors are written in a different order.'

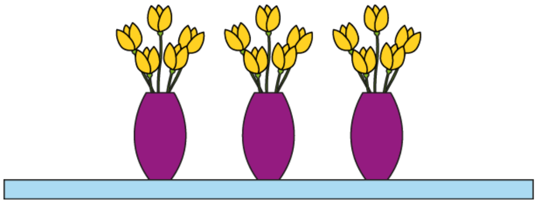
 2019 pilot

Commutativity of multiplication – 'one equation, two interpretations':

'Draw another picture of vases of flowers that could be represented by this equation.'

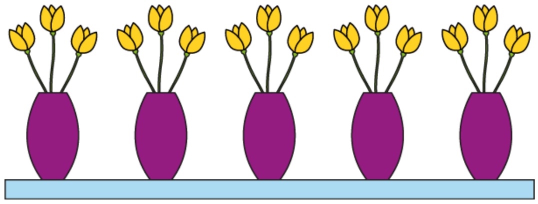
$$3 \times 5 = 15$$

A: original picture



- *'Three groups of five are equal to fifteen.'*

B: new picture



- *'Five groups of three are equal to fifteen.'*
- *'Three, five times is equal to fifteen.'*

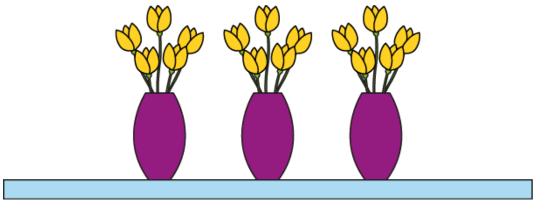
- *'Three groups of five is equal to five groups of three.'*

- *'What's the same?'*
  'Both pictures show fifteen flowers.'
- *'What's different?'*
  - *'Picture A shows three groups of five flowers.'*
  - *'Picture B shows five groups of three flowers (or three flowers, five times).'*

'Could both pictures also be represented by this equation?'

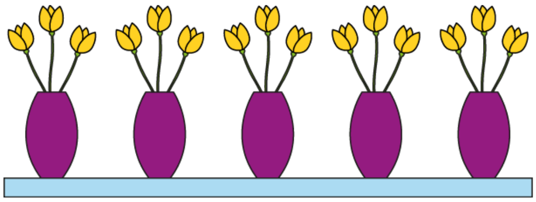
$$5 \times 3 = 15$$

A



- *'Three groups of five are equal to fifteen.'*
- *'Five, three times is equal to fifteen.'*

B



- *'Five groups of three are equal to fifteen.'*

- *'Three groups of five is equal to five groups of three.'*

- *'Decide whether each pair of expressions have the same product or not. Draw a picture, or use counters, to explain your answer in each case.'*

	Products equal	**Products *not* equal**
2×3 and 3×2		
5×4 and 4×5		
5×5 and 6×5		
12×3 and 13×2		
3×22 and 23×2		

1:2

Now present a related quotitive division problem, using the same context and values as in step *1:1*: *'I have fifteen flowers. If I put five flowers in each vase, how many vases do I need?'*

Record using missing-number multiplication calculations, asking children to describe what each number represents. Then present the pair of division calculations shown opposite and ask children to identify which is correct.

Once the correct equation has been identified, draw attention to the fact that the dividend has the same value as the product in the multiplication equations. Use the following generalised statement: ***'The <u>product</u> in the multiplication equation has the same value as the <u>dividend</u> in the matching division equation.'***

Finally, ask children to calculate the missing number to complete the three equations, and then reiterate that the value of the product has 'become' the value of the dividend.

Identifying the correct division equation:

'I have fifteen flowers. If I put five flowers in each vase, how many vases do I need?'

$\square \times 5 = 15$ $\qquad$ $5 \times \square = 15$

- *'Which calculation represents this problem?'*

$15 \div 5 = \square$ ✓

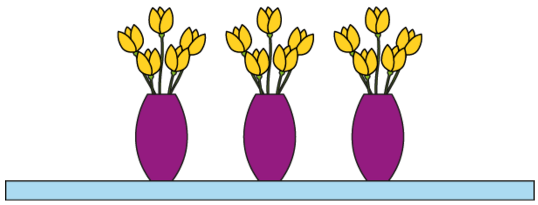
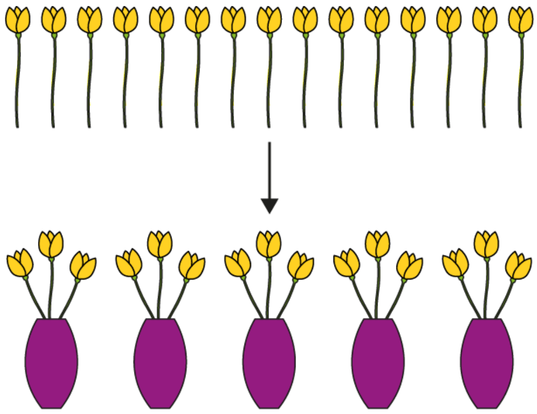
$5 \div 15 = \square$ ✗

Comparing the multiplication and division equations:

$\square \times 5 = \mathbf{15}$

$5 \times \square = \mathbf{15}$

$\mathbf{15} \div 5 = \square$

		Completing the equations: *'You need three vases.'*  $3 \times 5 = \mathbf{15}$ $5 \times 3 = \mathbf{15}$ $\mathbf{15} \div 5 = 3$
1:3	Now ask children to identify which division equation should be used if there are three flowers in each vase. As before, present the problem and write the missing-number multiplication equations. In this step, the focus is on how the factors in the multiplication equations correspond to the divisor and quotient in the division equation; so, before exploring the possible division equations, ask children to find the missing number in the multiplication equations. Then present the pair of division equations shown opposite and ask children to identify which is correct. Discuss the fact that both equations are 'correct', but that only one of the equations represents the division that needs to be performed to solve the problem.	Writing the multiplication equations: *'I have fifteen flowers. If I put three flowers in each vase, how many vases do I need?'*  $\boxed{5} \times 3 = 15$ $\quad 3 \times \boxed{5} = 15$ • *'You need five vases.'* Identifying the division calculation: *'Which calculation represents this problem?'* $15 \div 3 = 5$ ✓ Equation is correct; *does* represent the problem. $15 \div 5 = 3$ ✗ Equation is correct; but *15 ÷ 5* is *not* the calculation we need to perform.

1:4

Now compare the two problems and the corresponding equations from steps *1:2* and *1:3*). First, reiterate the generalisation from step 1:2: ***'The <u>product</u> in the multiplication equation has the same value as the <u>dividend</u> in the matching division equation.'*** Then, ask children to describe how the factors in the multiplication equation correspond to the terms in the matching division equation; work towards the following generalisation: ***'The factors in the multiplication equation have the same values as the divisor and the quotient in the matching division equation.'***

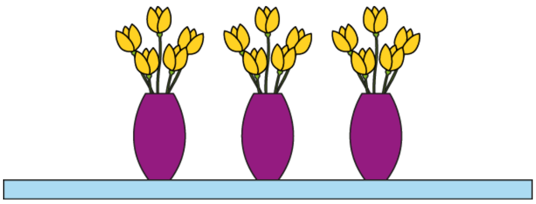
Draw children's attention to the fact that when we need to solve a problem, the known factor in the multiplication equation becomes the divisor and the missing factor becomes the quotient.

'I have fifteen flowers.'

'If I put five flowers in each vase, how many vases do I need?'

[3] × 5 = **15**   5 × [3] = 15

15 ÷ 5 [3]

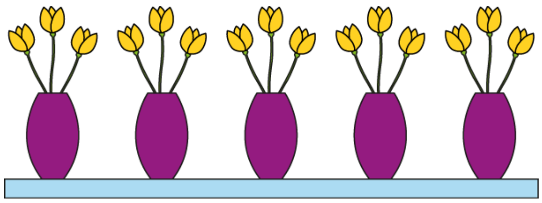


- *'You need three vases.'*

'If I put three flowers in each vase, how many vases do I need?'

[5] × 3 = **15**   3 × [5] = 15

15 ÷ 3 [5]



- *'You need five vases.'*

Summary:

factor × ? = **product**   ? × factor = **product**

↓

dividend ÷ divisor = quotient

1:5

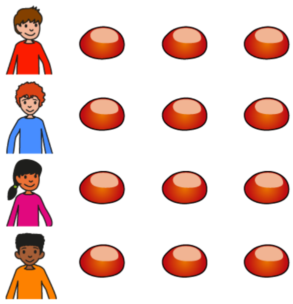
Now repeat, using a partitive division context such as that shown below. The presentation of the conkers resembles an array, with two grouping interpretations.

'There are twelve conkers.'

'If the conkers are shared equally between <u>four</u> children, how many conkers does each child get?'

$\boxed{3} \times 4 = \mathbf{12}$ $\quad 4 \times \boxed{3} = 12$

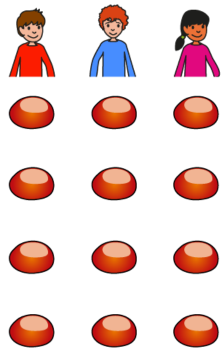
$\mathbf{12} \div 4 = \boxed{3}$



'If the conkers are shared equally between <u>three</u> children, how many conkers does each get?'

$\boxed{4} \times 3 = \mathbf{12}$ $\quad 3 \times \boxed{4} = 12$

$\mathbf{12} \div 3 = \boxed{4}$



1:6

Also explore multiplication equations where zero is a factor, and how this translates into the corresponding division equation. In segment *2.4 Times tables: groups of 10 and of 5, and factors of 0 and 1*, children generalised about the product when one factor is zero: ***'When zero is a factor, the product is zero.'***

Remind children of this, referring to some known multiplication facts.

Then work systematically, as shown opposite and on the next page, to look at the types of division expressions/equations we can write with zero in them. For each possibility (e.g. *0 ÷ 3*, *3 ÷ 0* and *0 ÷ 0*), ask children:

- to write the corresponding missing-factor multiplication equation (reminding children that the product

- *'Is it okay to write this division calculation?'*

  $0 \div 3 = \boxed{}$

- *'Write this calculation as a multiplication equation with a missing factor.'*

  $\boxed{} \times 3 = 0 \quad 3 \times \boxed{} = 0$

- *'Can we solve the multiplication equations?'*
  'Yes'

  $\boxed{0} \times 3 = 0$ ✓ $\quad 3 \times \boxed{0} = 0$ ✓

- *'So is it okay to write this division calculation?'*

  $0 \div 3 = \boxed{}$

  'Yes.'

  $0 \div 3 = 0$ ✓

	always 'becomes' the dividend) • to reason about whether the corresponding multiplication equation makes sense and/or can be solved • to then decide whether we should or shouldn't write a division calculation of this type. For the equation $0 \div 3 = 0$, you can tell a story to demonstrate that it makes sense, for example: *'There are no apples; the apples are shared between three people; the people don't get any apples.'* Make the following generalisation: '***We should never write a calculation where the divisor is zero.'*** Note, the equation $0 \div 0 = ?$ isn't *incorrect*, but children are discouraged from writing such an equation since it isn't *useful*; there are an infinite number of solutions, so we can't solve any problems with this equation (for example, whole number solutions would be $0 \div 0 = 1, 0 \div 0 = 2, 0 \div 0 = 3\ldots$).	• *'Is it okay to write <u>this</u> division calculation?'* $3 \div 0 = \square$ • *'Write this calculation as a multiplication equation with a missing factor.'* $\square \times 0 = 3$ $\quad 0 \times \square = 3$ • *'Do these multiplication equations make sense?'* *'No. Because one factor is zero, the product must be zero.'* $\square \times 0 = 3$ ✗ $\quad 0 \times \square = 3$ ✗ • *'So is it okay to write this division calculation?'* $3 \div 0 = \square$ *'No.'* • *'Is it okay to write <u>this</u> division calculation?'* $0 \div 0 = \square$ • *'Write this calculation as a multiplication equation with a missing factor.'* $\square \times 0 = 0$ $\quad 0 \times \square = 0$ • *'Can we solve these calculations?'* *'Yes. But the answer could be any number.'* $\square \times 0 = 0$? $\quad 0 \times \square = 0$? • *'So is it okay to write this division calculation?'* $0 \div 0 = \square$ *'No. This isn't a useful equation.'*

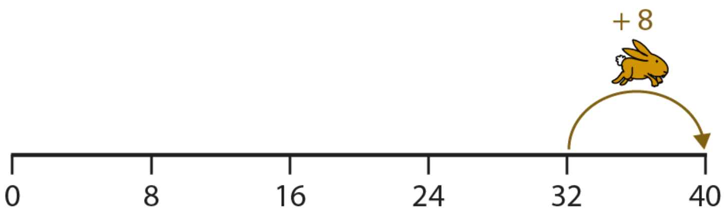
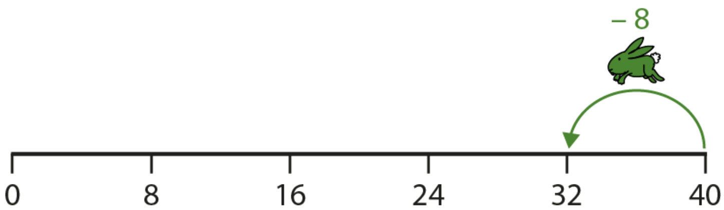
1:7	Now solve some division calculations of the form *0 ÷ n = ?*, using the corresponding multiplication facts for support, as shown opposite. Work towards the generalisation: ***'When the dividend is zero, the quotient is zero.'***	$\square \times 1 = 0$ $0 \div 1 = \square$ $\square \times 2 = 0$ $0 \div 2 = \square$ $\square \times 3 = 0$ $0 \div 3 = \square$ $\square \times 4 = 0$ $0 \div 4 = \square$ $\square \times 5 = 0$ $0 \div 5 = \square$ $\square \times 10 = 0$ $0 \div 10 = \square$
1:8	To complete this teaching point, provide children with practice matching/writing associated multiplication and division equations.	*'Draw lines to connect each multiplication equation to a matching division equation.'* $15 = 3 \times 5$ $24 \div 6 = 4$ $24 = 8 \times 3$ $0 \div 7 = 0$ $22 = 2 \times 11$ $15 \div 3 = 5$ $0 = 7 \times 0$ $7 \div 7 = 1$ $24 = 12 \times 2$ $11 = 22 \div 2$ $7 = 7 \times 1$ $24 \div 8 = 3$ $4 \times 6 = 24$ $2 = 24 \div 12$ Dòng nǎo jīn: • *'Write two division equations using this multiplication equation.'* $15 \times 4 = 60$ • *'Use the following equation to help you solve the problem below about Jake's pocket money.'* $75 \times 5 = 375$ *'Jake gets the same amount of pocket money each day from Monday to Friday. At the end of the week Jake has £3.75.'* *'Write a division equation to show how much money Jake gets each day.'*

Teaching point 2:

Multiplication is distributive: multiplication facts can be derived from related known facts by partitioning one of the factors, and this can be interpreted as partitioning the number of groups; two-part problems that involve addition/subtraction of products with a common factor can be efficiently solved by applying the distributive law.

Steps in learning

	Guidance	**Representations**

2:1	Children have already used the distributive law informally. In particular, for each times table learnt, children have noticed and applied the fact that adjacent multiples of a number have a difference equal to that number (e.g. $5 \times 2 = 4 \times 2 + 2$ and $4 \times 2 = 5 \times 2 - 2$).
	In this teaching point, knowledge of the distributive law is formalised and extended, including:
	• how known multiplication facts can be used to derive other products, by adding or subtracting either one group or several groups (partitioning the number of groups), for example:
	  ◦ five groups of eight is equal to three groups of eight plus two groups of eight
	  ◦ three groups of eight is equal to five groups of eight minus two groups of eight
	(steps *2:1–2:5*)
	• how we can apply this understanding, irrespective of the context of the problem we are solving (whether the 'common' factor represents the number of groups, the group size, or neither in the case of an abstract problem), given our knowledge of commutativity (for more, see the *Overview of Learning*)
	(steps *2:6–2:8*)
		Adding one multiple:
		
		
		$5 \times 8 = 4 \times 8 + 8$
		Subtracting one multiple:
		
		
		$4 \times 8 = 5 \times 8 - 8$

	• how the distributive law can be used to simplify calculations that involve the addition/subtraction of products that share a common factor; for example, if a problem requires us to calculate $9 \times 4 + 9 \times 5$, it is not necessary to perform both multiplications and add the product; instead we can apply the distributive law and calculate 9×9. (steps *2:9–2:10*) Begin by reviewing children's knowledge of the 'adjacent multiples rule', as exemplified on the previous page. Use the familiar representation of a rabbit jumping forwards or backwards one multiple on the number line, and write the associated equations in the form: • $5 \times 8 = 4 \times 8 + 8$ • $4 \times 8 = 5 \times 8 - 8$ Although this should be very familiar to children by now, it is useful to begin here, as the next steps will build on this knowledge. Note that the equation $8 = 5 \times 8 - 4 \times 8$ is also valid, but does not match the number line representations shown on the previous page.	
2:2	Now represent the same calculations using a part–part–whole diagram (cherry diagram) with eight-value counters, using it to write out the more 'expanded' form of the equations (including the factor of '1'): • $5 \times 8 = 4 \times 8 + 1 \times 8$ • $4 \times 8 = 5 \times 8 - 1 \times 8$ Also use stacked number lines, as shown on the next page, to reinforce the link between the known additive fact and application of the distributive law.	

Use the following stem sentences (as exemplified opposite):

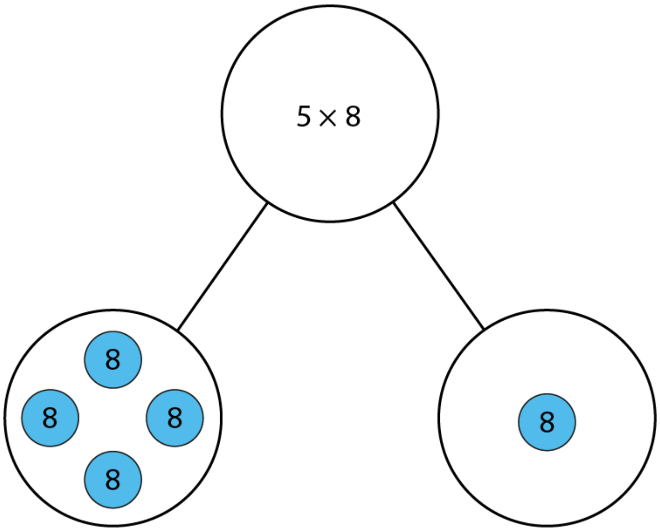
- ***'___ is equal to ___ plus ___, so ___ times ___ is equal to ___ times ___ plus ___ times ___.'***
- ***'___ is equal to ___ minus ___, so ___ times ___ is equal to ___ times ___ minus ___ times ___.'***

Important points to note about these stem sentences are:

- the language of '___ *times* ___' is used instead of '___ *groups of* ___' to support children in moving towards working solely with the equations and solving contextual problems irrespective of whether the 'common' factor represents the number of groups or the group size
- the language represents unitising in the 'common' factor (eight, in the example opposite) and follows the same pattern as that used, for example, in *Spine 1: Number, Addition and Subtraction*, segment *1.8* where children were unitising in tens.

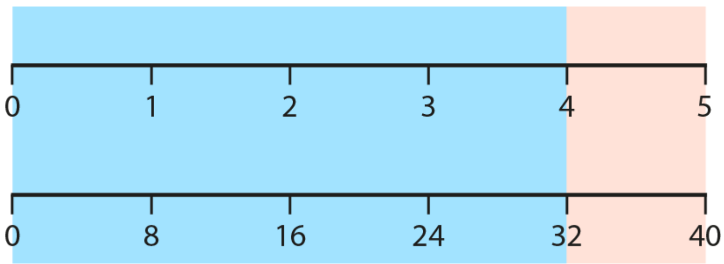
A third equation can be written based on the representations opposite ($1 \times 8 = 5 \times 8 - 4 \times 8$). This has not been included since it is not particularly useful in terms of deriving products from known facts. However, you could challenge children to write the third equation using the representations for support.

Linking addition/subtraction of one multiple to partitioning – part–part–whole and equations:



$\mathbf{5} = \mathbf{4} + \mathbf{1}$ $\mathbf{5} \times 8 = \mathbf{4} \times 8 + \mathbf{1} \times 8$ $= 32 + 8$ $= 40$ *'Five is equal to four plus one, so five times eight is equal to four times eight plus one times eight.'*	$\mathbf{4} = \mathbf{5} - \mathbf{1}$ $\mathbf{4} \times 8 = \mathbf{5} \times 8 - \mathbf{1} \times 8$ $= 40 - 8$ $= 32$ *'Four is equal to five minus one, so four times eight is equal to five times eight minus one times eight.'*

Linking addition/subtraction of one multiple to partitioning – stacked number lines:



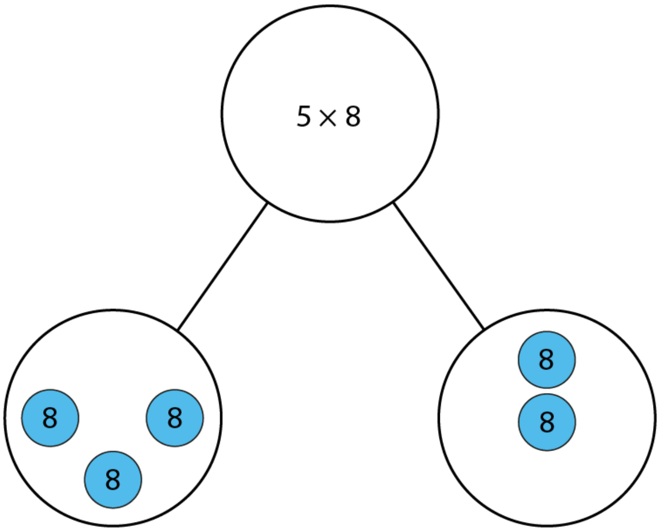
2:3

Continuing with a total of five eights, now move one of the counters across on the part–part–whole diagram, so that the five eights are now partitioned into three eights and two eights. Write the corresponding equations and represent them on a number line. Use the stem sentence from step *2:2* to describe the calculations.

As in step *2:2*, only two equations have been written based on the representations. You can, again, challenge children to write the third equation:

$2 \times 8 = 5 \times 8 - 3 \times 8$

Linking addition/subtraction of *two multiples* to partitioning – part–part–whole and equations:



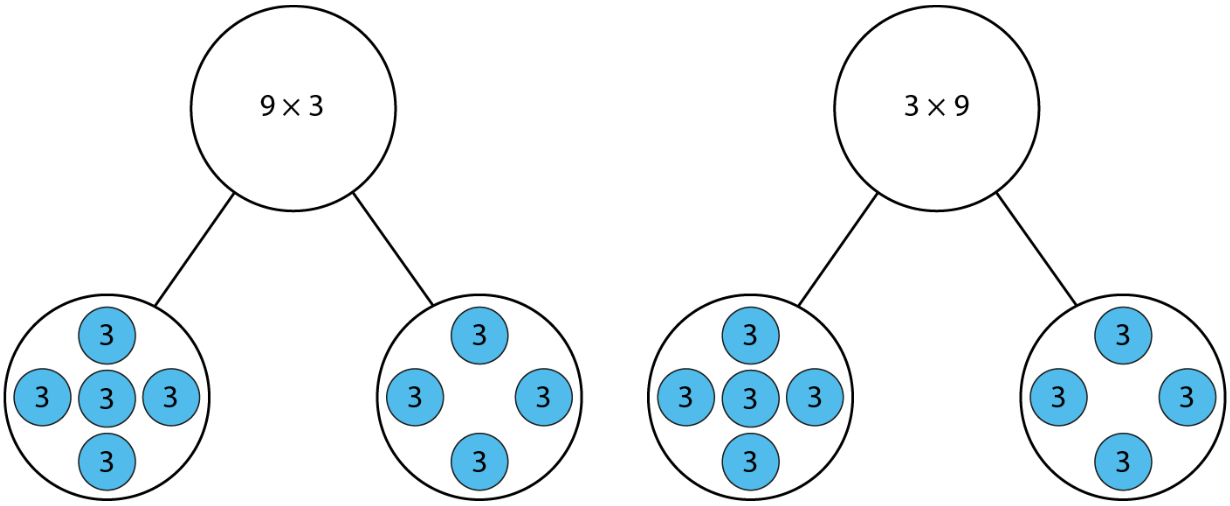
$\mathbf{5} = \mathbf{3} + \mathbf{2}$	$\mathbf{3} = \mathbf{5} - \mathbf{2}$
$\mathbf{5} \times 8 = \mathbf{3} \times 8 + \mathbf{2} \times 8$	$\mathbf{3} \times 8 = \mathbf{5} \times 8 - \mathbf{2} \times 8$
$= 24 + 16$	$= 40 - 16$
$= 40$	$= 24$
'Five is equal to three plus two, so five times eight is equal to three times eight plus two times eight.'	*'Three is equal to five minus two, so three times eight is equal to five times eight minus two times eight.'*

Linking addition/subtraction of *two* multiples to partitioning – stacked number lines:

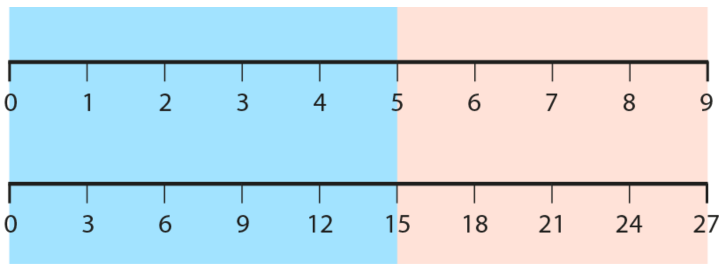
2:4 Use the same representations to look at cases with other group sizes; for example, *'Seven is equal to five plus two, so seven times three is equal to five times three plus two times three.'*

Also include some examples where the factors are written in the other order (with the group size first), amending the stem sentence accordingly, as exemplified below. Note that the example provided is not the most efficient strategy for calculating *9 × 3*. However, for now the focus is on the structure of the distributive law rather than on its applications.

Varying the order of the factors:



$\mathbf{9} = \mathbf{5} + \mathbf{4}$ $3 \times \mathbf{9} = 3 \times \mathbf{5} + 3 \times \mathbf{4}$ $= 15 + 12$ $= 27$ *'Nine is equal to five plus four, so three times nine is equal to three times five plus three times four.'*	$\mathbf{5} = \mathbf{9} - \mathbf{4}$ $3 \times \mathbf{5} = 3 \times \mathbf{9} - 3 \times \mathbf{4}$ $= 27 - 12$ $= 15$ *'Five is equal to nine minus four, so three times five is equal to three times nine minus three times four.'*

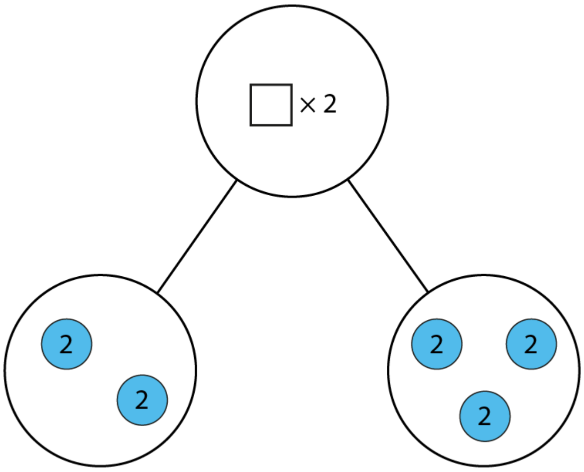


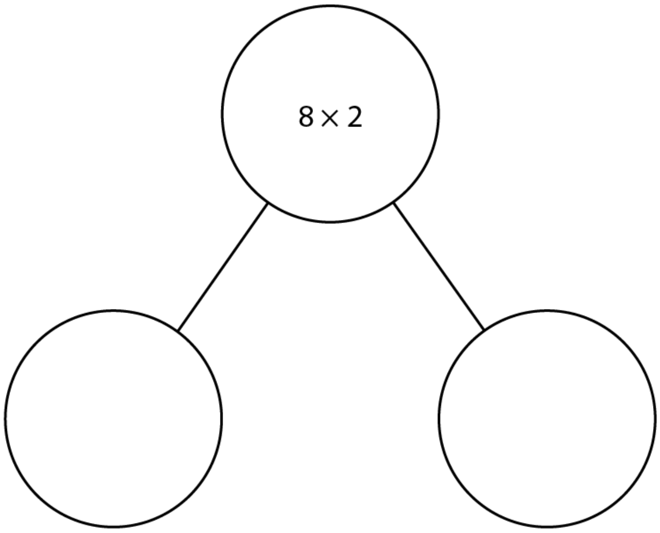
<table>
<tr>
<td>2:5</td>
<td>
At this point, provide children with some practice partitioning the number of groups and writing associated equations, including:

completing an expression to represent the ‘whole’ in part–part–whole diagrams
working practically to distribute counters in the ‘parts’ on part–part–whole diagrams, and writing the associated equations; this is exemplified opposite (the second part–part–whole diagram) – note that since either factor can represent the group size you should ensure that children can show both interpretations, for example:

8×2 can represent eight groups of two, which could be partitioned, for example, as $6 \times 2 + 2 \times 2$
8×2 can represent eight twice (two groups of eight), which could be partitioned as $8 \times 1 + 8 \times 1$

true/false problems
writing equations to represent contextual problems.

Encourage children to continue using the stem sentence to support unitising (in units of the ‘common’ factor).
</td>
<td>
Completing the whole:
‘Fill in the missing number.’


Distributing counters:
‘Use counters to represent the parts. For each solution you find, write three equations.’


$8 \times 2 = \square \times \square + \square \times \square$
$\square \times \square = 8 \times 2 - \square \times \square$
$\square \times \square = 8 \times 2 - \square \times \square$
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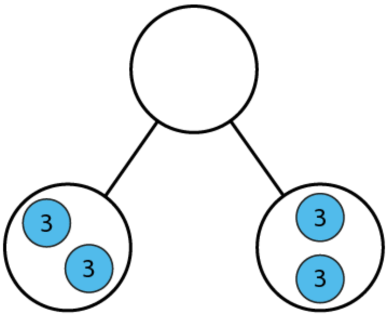
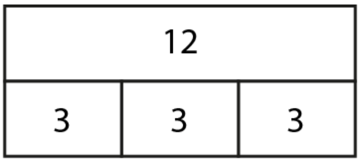
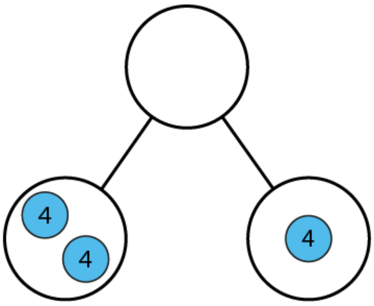
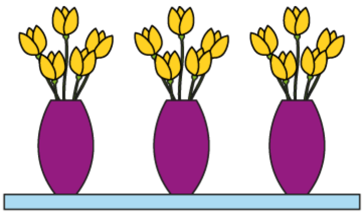
</table>

True/false problems:

Does this show *4* × *3*?	**✓ or ✗**
	
$3 \times 3 + 1 \times 3$	
	
	
	
$2 \times 4 + 4 \times 1$	

<table>
<tr>
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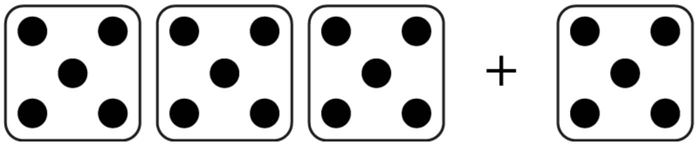
<td></td>
<td>Contextual problem:
• *'Write an equation to represent Eloise's dice rolls.'*
Eloise rolls 3 fives. Then she rolls 1 more five.

• *'Write an equation to represent the remaining number of legs after two of the spiders have crawled away.'*
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<tr>
<td>2:6</td>
<td colspan="2">Now, through the use of arrays and the familiar stem sentence, demonstrate that the distributive law can be applied to any multiplication equation, irrespective of whether the 'common' factor represents the number of groups or the group size. Note that it is not necessary to consider whether we are partitioning the number of groups (as in steps *2:1–2:5*) or the group size (which can be more difficult for children to access); in all cases, known additive composition (for example, eight can be made up of five and three), is applied to partition one of the factors (e.g. $6 \times 8 = 6 \times 5 + 6 \times 3$).

Useful examples to consider include cases where:
• six is a factor (partitioning into five and one)
• seven is factor (partitioning into five and two)
• nine is a factor (using the fact that $9 = 10 - 1$).

Note that children have already seen array representations of these relationships in the relevant times tables segments; however, now the equations are being written out 'in full'. For each example given below, factors are written out both ways for completeness, but the order of the factors does not imply that either of them is interpreted as either the number of groups or the size of the groups.

Work through a range of examples, gradually removing the array representation until children can confidently work with the equations alone.</td>
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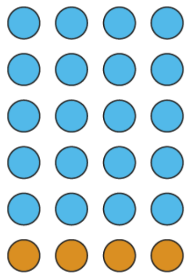
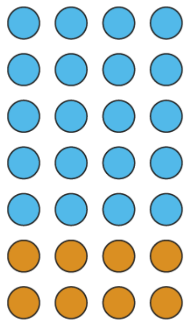
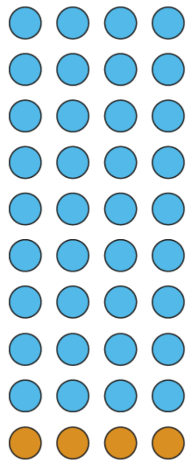
Example 1 –'6' is a factor:	Example 2 – '7' is a factor:	Example 3 – '9' is a factor:
		
$\mathbf{6} = \mathbf{5} + \mathbf{1}$ $4 \times \mathbf{6} = 4 \times \mathbf{5} + 4 \times \mathbf{1}$ $= 20 + 4$ $= 24$ *'Six is equal to five plus one, so four times six is equal to four times five plus four times one.'* or $\mathbf{6} = \mathbf{5} + \mathbf{1}$ $\mathbf{6} \times 4 = \mathbf{5} \times 4 + \mathbf{1} \times 4$ $= 20 + 4$ $= 24$ *'Six is equal to five plus one, so six times four is equal to five times four plus one times four.'*	$\mathbf{7} = \mathbf{5} + \mathbf{2}$ $4 \times \mathbf{7} = 4 \times \mathbf{5} + 4 \times \mathbf{2}$ $= 20 + 8$ $= 28$ *'Seven is equal to five plus two, so four times seven is equal to four times five plus four times two.'* or $\mathbf{7} = \mathbf{5} + \mathbf{2}$ $\mathbf{7} \times 4 = \mathbf{5} \times 4 + \mathbf{2} \times 4$ $= 20 + 8$ $= 28$ *'Seven is equal to five plus two, so seven times four is equal to five times four plus two times four.'*	$\mathbf{9} = \mathbf{10} - \mathbf{1}$ $4 \times \mathbf{9} = 4 \times \mathbf{10} - 4 \times \mathbf{1}$ $= 40 - 4$ $= 36$ *'Nine is equal to ten minus one, so four times nine is equal to four times ten minus four times one.'* or $\mathbf{9} = \mathbf{10} - \mathbf{1}$ $\mathbf{9} \times 4 = \mathbf{10} \times 4 - \mathbf{1} \times 4$ $= 40 - 4$ $= 36$ *'Nine is equal to ten minus one, so nine times four is equal to ten times four minus one times four.'*

2:7	Apply the learning from step *2:6* to a contextual problem where the most efficient method of solving it corresponds to partitioning the factor that represents the *group size*. For example, with the problem opposite, some children may, for now, not be fluent with the fact 8×7, and as such would like to use the distributive law to help them perform the calculation by partitioning seven, even though the problem is about groups of seven, not seven groups. Use the following approach: • Interpret the problem and write a multiplication expression to represent it. • Apply the distributive law to perform the calculation. • Interpret the answer in the context of the problem. (Note: the factors can be written in either order; only one option is exemplified opposite.)	*'Felicity's summer holiday is eight weeks long. How many days is this?'* • *'Interpret the problem:'* 8×7 • *'Apply the distributive law:'* $7 = 5 + 2$ so $8 \times 7 = 8 \times 5 + 8 \times 2$ $= 40 + 16$ $= 56$ • *'Interpret the answer:'* Felicity's holiday is 56 days long.

2:8 At this point, provide children with practice using the distributive law, including:

- missing-number/symbol problems
- completing ratio charts (for the examples below, encourage children to use their one, two and five times-table facts, along with addition and subtraction, rather than using their six and seven times tables)
- contextual problems.

Missing number/symbol problems:

'Fill in the missing numbers and symbols.'

$3 \times 7 \bigcirc 3 \times 2 + 3 \times 5$

$4 \times 7 \bigcirc 4 \times 5 + 4 \times 2$

$4 \times 7 \bigcirc 4 \times 6 + 4 \times 1$

$5 \times 7 > \square \times 2 + \square \times 5$

$6 \times 7 < 5 \times \square + 5 \times \square$

$8 \times 6 \bigcirc 8 \times 5 + 8 \times 1$

$8 \times 6 \bigcirc 8 \times 4 + 8 \times 1$

$8 \times 6 \bigcirc 8 \times 4 + 8 \times 2$

$8 \times 6 \bigcirc 8 \times 4 + 8 \times 3$

$8 \times 6 \bigcirc 8 \times 3 + 8 \times 3$

$12 \times 5 \bigcirc 12 \times 8 - 12 \times 2$

$11 \times \square = 11 \times 5 + 11 \times 4$

Completing ratio charts:

'Fill in the missing numbers in these ratio charts.'

	× 2	**× 5**	**× 7**
0	0	0	0
1	2	5	7
2	4	10	14
3	6	15	21
4	8	20	28
5	10	25	35
6	12	30	42
7	14	35	
8	16	40	
9	18	45	
10	20	50	
11	22	55	
12	24	60	

	× 5	**× 1**	**× 6**
0	0	0	0
1	5	1	
2			
3	15		
		4	
5			30
6	30		
7		7	
8			
9			54
10			60
11			
	60		

Contextual problem:

'There are some 5 p and 2 p coins. How much money is there altogether?'

Dòng nǎo jīn:

- *'Evie writes this in her book:'*

  3 = 2 + 1

so

$$9 \times 3 = 9 \times 2 + 9 \times 1$$
$$= 18 + 1$$
$$= 19$$

  'Is her calculation correct?'

- *'Fill in the missing number.'*

  $9 \times 7 = 5 \times 9 + \square \times 9$

Dòng nǎo jīn:

'Some children are calculating seven times nine.'

Child A

$$7 \times 9 = 7 \times 10 - 7 \times 1$$
$$= 70 - 7$$
$$= 63$$

Child B

$$7 \times 9 = 9 \times 7$$
$$9 \times 7 = 9 \times 5 + 9 \times 2$$
$$= 45 + 18$$
$$= 63$$

Child C

$$7 \times 9 = 7 \times 10 - 7$$
$$= 70 - 7$$
$$= 63$$

'Who is correct? Whose method do you prefer? Why?'

2:9

To complete this teaching point, explore how two-part problems with a 'common' factor (e.g. $9 \times 4 + 9 \times 5$) can be most efficiently solved by applying the distributive law and performing one multiplication calculation rather than two multiplication calculations and an addition (for example, calculating 9×9 rather than calculating and adding together 9×4 and 9×5).

Begin by presenting a calculation involving the *addition* of two products. Work through the calculation as a class, finding both products and then adding them together. Then solve the problem again by applying the distributive law so that only one multiplication

Addition of two products:

9 × 4 + 9 × 5 = ?	
Method 1 – calculate the products and add	Method 2 – apply the distributive law
$9 \times 4 = 36$ $9 \times 5 = 45$ $36 + 45 = 81$ so $\mathbf{9 \times 4 + 9 \times 5 = 81}$	$9 \times 4 + 9 \times 5 = 9 \times 9$ $9 \times 9 = 81$ so **9 × 4 + 9 x 5 = 81**

<table>
<tr>
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<td>calculation needs to be performed. Compare the two methods and ask children which they think is best and why. Encourage them to notice that applying the distributive law is more efficient; discuss how this approach is quicker and how there are fewer opportunities for error.

Then repeat for a calculation involving subtraction of one product from another.</td>
<td>Subtraction of one product from another:
<table>
<tr><th colspan="2">3 × 7 – 3 × 5 = ?</th></tr>
<tr><td>Method 1 – calculate the products and add

3 × 7 = 21
3 × 5 = 15
21 – 15 = 6
so
3 × 7 – 3 × 5 = 6</td><td>Method 2 – apply the distributive law

3 × 7 – 3 × 5 = 3 × 2
3 × 2 = 6

so
3 × 7 – 3 × 5 = 6</td></tr>
</table>
</td>
</tr>
<tr>
<td>2:10</td>
<td colspan="2">Finally, provide children with practice solving two-part contextual problems with a ‘common’ factor (including measures contexts). Work through the first example as a class, encouraging children to read the entire question carefully, and then to write a single equation to represent the whole problem. Children should then be able to see how they can apply the distributive law so they need to perform only one multiplication calculation.

Include questions:

that correspond to partitioning the number of groups; the factor that remains the same in both multiplications represents the group size
that correspond to partitioning the group size; the factor that remains the same in both multiplications represents the number of groups.

Example word problems:

‘Simon has two sheets of stickers, with six stickers on each sheet. Cali also has two sheets of stickers, but she has only four stickers on each sheet.’

‘How many stickers do they have altogether?’
‘How many more stickers does Simon have than Cali?’

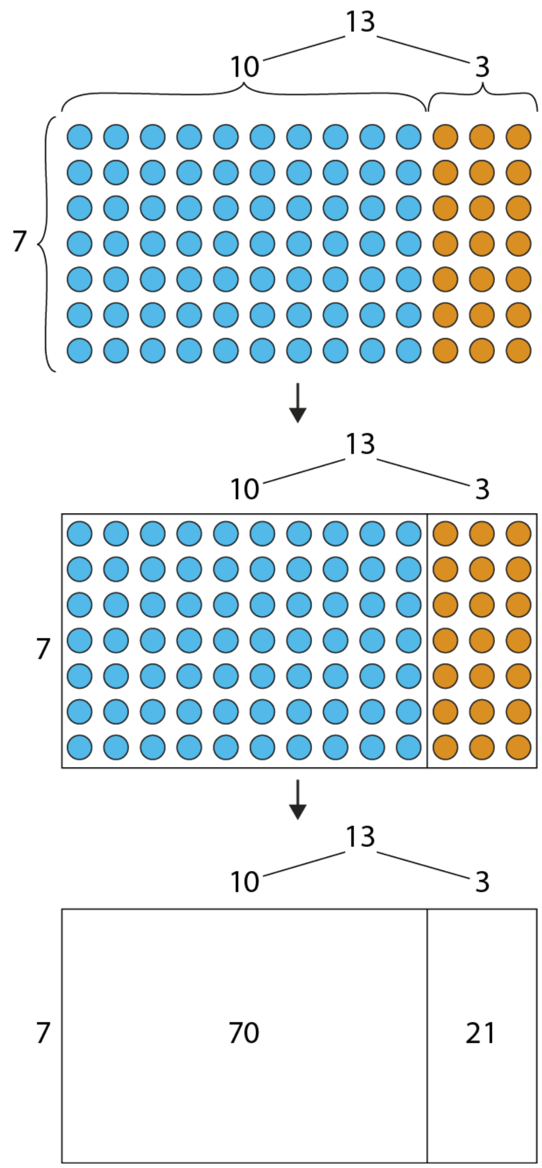
‘Last week, Sam ran five kilometres each day on Monday, Tuesday and Wednesday. This week she ran five kilometres on both Monday and Tuesday. How far has she run altogether?’
‘The school cook has eight ten-kilogram bags of potatoes. He uses three of the bags. How many kilograms of potatoes does he have left?’
‘Jemima saved £6 a week for five weeks, then £4 a week for the next five weeks. How much did she save in total?’

</td>
</tr>
</table>

Teaching point 3:

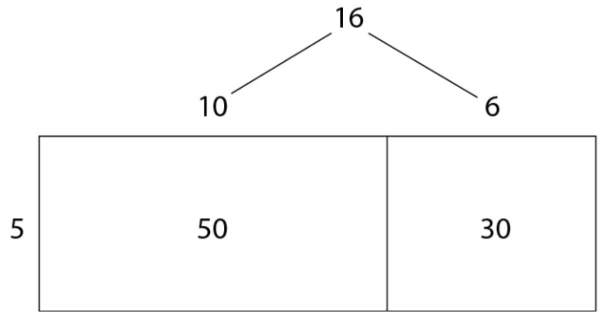
The distributive law can be used to derive multiplication facts beyond known times tables.

Steps in learning

	Guidance	Representations
3:1	This teaching point applies the distributive law to calculate products involving numbers beyond known times-table facts. This serves as preparation for the next segment, in which the eleven and twelve times tables will be built up, systematically, using the distributive law. The examples in this teaching point are restricted to multiplication of a single-digit number by a teen number, for example *7 × 13* or *13 × 7*. Note that an array representation, leading to a 'grid method' is used, and it is not necessary to connect one of the numbers with group size and the other with the number of groups; having said that, the strategy could be applied to contextual problems with either structure (for example, seven groups of thirteen, or thirteen groups of seven). The array/grid should *not* be used as a tool for calculating the answer; rather, it should be used to expose the mathematical structure and demonstrate how the distributive law can be applied. A sketch of a rectangle is an interim scaffold between drawing the array and just using an equation (see *Example 2* on the next page). Note that children will formally explore the connection between area and multiplication in segment *2.16 Multiplicative contexts: area and perimeter 1*. Begin by reminding children about the 'ten-and-a-bit' structure of the teen numbers (see *Spine 1: Number, Addition and Subtraction*, segment *1.10*). A quick	Example 1 – with array/grid method:  $7 \times 13 = 7 \times 10 + 7 \times 3$ $= 70 + 21$ $= 91$

<table>
<tr>
<td></td>
<td>count through the teen numbers using a Gattegno chart could be used to illustrate this.

Then present a calculation such as 7 × 13, and demonstrate how the distributive law can be applied by partitioning the teen number into ten and ones. Use the array/grid for support, as shown on the previous page, alongside equations.

Work through more examples like this, gradually removing the array/grid scaffold and moving to a more concise set of equations. Vary the order of the factors, using a mixture of two-digit × one-digit and one-digit × two-digit examples.</td>
<td>Example 2 – less scaffolding, factor order varied:


$5 \times 16 = 5 \times 10 + 5 \times 6$
$= 50 + 30$
$= 80$

Example 3 – no scaffolding:
$6 \times 18 = 6 \times 10 + 6 \times 8$
$= 60 + 48$
$= 108$</td>
</tr>
<tr>
<td>3:2</td>
<td>Now explore alternative ways of applying the distributive law to such problems, for example:
for larger teen numbers, consider multiplying by twenty and then subtracting; note that children have not yet learnt to multiply by a multiple of ten, though this is another valid strategy that children can apply after segment 2.13 Calculation: multiplying and dividing by 10 or 100
partitioning even teen numbers into two equal parts, using halving and doubling strategies; discuss which method children prefer; they might make observations such as 'I like partitioning into ten and ones best because it's easy to multiply by ten' or 'I like partitioning into two equal parts because you only have to use one multiplication fact, then you can double.'</td>
<td>Working flexibly and efficiently – example 1:
<table><tr><th colspan="2">6 × 18 = ?</th></tr><tr><td>$6 \times 18 = 6 \times 10 + 6 \times 8$
$= 60 + 48$
$= 108$</td><td>$6 \times 18 = 6 \times 20 - 6 \times 2$
$= 120 - 12$
$= 108$</td></tr></table>
Working flexibly and efficiently – example 2:
<table><tr><th colspan="2">7 × 12 = ?</th></tr><tr><td>$7 \times 12 = 7 \times 10 + 7 \times 2$
$= 70 + 14$
$= 84$</td><td>$7 \times 12 = 7 \times 6 + 7 \times 6$
$= 42 + 42$
$= 84$</td></tr></table></td>
</tr>
</table>

	The aim isn't to choose one 'best method', but to encourage children to get into the habit of examining the numbers involved in a calculation and working flexibly to solve it using an efficient method. Of course, there are even more ways to solve each of the calculations given on the previous page and opposite, including further alternative partitioning of the two-digit number; for *Example 3*, an alternative strategy would be *9 × 14 = 10 × 14 – 14* (again, this can be considered after children have the tools to multiple two-digit numbers by ten in segment *2.13*).	Working flexibly and efficiently – example 3: **9 × 14 = ?** 9 × 14 = 9 × 10 + 9 × 4 = 90 + 36 = 126 9 × 14 = 9 × 7 + 9 × 7 = 63 + 63 = 126

3:3	Provide children with varied practice applying the distributive law to calculations beyond their known times-table facts, including:
	• missing-number problems
	• real-life problems including measures contexts, for example:
	  ◦ *'A large bucket can hold fourteen litres of water. How much water would eight full buckets hold?'*
	  ◦ *'Danior spends four hours each week practising the piano. How many hours of practice does he do in sixteen weeks?'*
	  ◦ Dòng nǎo jīn:
	'Libby has a ten-litre bucket and a five-litre bucket. She uses them both, six times, to fill a container with water.
	Jen only has a ten-litre bucket. How many times will Jen need to use her bucket to fill the same size container as Libby?'
		Missing-number problem:
		'Fill in the missing numbers.'
		
		5 × 14 = □ × 10 + □ × 4
		= □ + □
		= □
		
		4 × 17 = □
		4 × 18 = □
		4 × 19 = □

