MathsHUBS Resources

National Centre for Excellence in the Teaching of Mathematics

Mastery Professional Development Materials

2 Operating on number

Theme overview

Guidance document | Key Stage 3

Making connections

'Teaching for mastery' describes the elements of classroom practice and school organisation that combine to give students the best chance of developing a deep, connected, embedded and sustainable understanding of mathematics.

At any one point in a student's journey through school, achieving mastery means acquiring a secure understanding of the mathematics that has been taught to enable them to move on to more advanced material.

To achieve this, students need to understand the interconnected nature of mathematics and how one idea builds on and develops from other ideas. To this end, the NCETM has identified a set of six 'mathematical themes' within Key Stage 3 mathematics that bring together a group of connected ideas or 'core concepts'.

The theme *Operating on number* covers the following interconnected core concepts:

2.1 Arithmetic procedures

2.2 Solving linear equations

Please note that these materials are principally for professional development purposes. Unlike a textbook scheme they are not designed to be directly lifted and used as teaching materials. The materials can support teachers to develop their subject and pedagogical knowledge and so help to improve mathematics teaching in combination with other high-quality resources, such as textbooks.

Why is this mathematical theme important?

The ability to calculate is a fundamental skill in mathematics and this, of course, includes the requirement that students know and use standard methods of calculation. However, students who know these methods solely as a set of memorised steps, without any understanding of why they work and the laws of arithmetic on which they are based, may quickly forget them.

This theme is about students understanding the structures underpinning calculation using each of the four operations. The aim is that students are able to calculate efficiently and flexibly, using a range of strategies with all types of numbers (including decimals and fractions) and in a variety of problem-solving situations.

For students to have a sense of the mathematics curriculum as a connected whole rather than a series of disconnected 'topics' with their own different rules and formulae, it is important to appreciate that number and algebra are connected. Solving equations is essentially concerned with operations on, as yet, unknown numbers, and so has also been included in this theme.

Key underpinning knowledge

Several important considerations are key to students gaining a secure and deep understanding of the mathematics within this theme, namely:

- that methods of addition and subtraction are based on the idea that we can add and subtract 'units' of the same value (i.e. adding tens to tens; hundreds to hundreds; tenths to tenths, etc.) and that this, together with our place-value system, allows for efficient methods of calculation
- that this concept of 'unitising' extends to the addition and subtraction of fractions (i.e. we can add or subtract fractions of the same denominator)
- that multiplication is not only repeated addition but also scaling, and that this image is vital for understanding the multiplication and division of fractions
- that division is not only sharing (or partitioning) but also repeated subtraction (e.g. 480 ÷ 4 would be thought of more easily as *'480 partitioned into four equal parts'*, whereas 300 ÷ 50 would more likely be thought of as *'How many 50s are in 300?'*). Such flexibility in thinking about division is helpful when first encountering dividing by fractions, e.g. $4 \div \frac{1}{5}$
- that fluency in calculating includes recognising (particularly in new and unfamiliar situations) what operations are required, knowing what properties of number and operations we need to exploit to arrive at a solution, and being able to interpret the solution appropriately
- that linear equations are essentially the formulation of a series of operations on, as yet, unknown numbers (perhaps represented by a letter symbol) and the solving of such equations is concerned with the undoing of such operations.

Statements of knowledge, skills and understanding

Each of the two core concepts within the theme *Operating on number* has been broken down further into a set of statements of knowledge, skills and understanding, as listed below.

2.1 Arithmetic procedures

2.1.1 Understand and use the structures that underpin addition and subtraction strategies

2.1.2 Understand and use the structures that underpin multiplication and division strategies

2.1.3 Know, understand and use fluently a range of calculation strategies for addition and subtraction of fractions

2.1.4 Know, understand and use fluently a range of calculation strategies for multiplication and division of fractions

2.1.5 Use the laws and conventions of arithmetic to calculate efficiently

2.2 Solving linear equations

2.2.1 Understand what is meant by finding a solution to a linear equation with one unknown

2.2.2 Solve a linear equation with a single unknown on one side where obtaining the solution requires one step

2.2.3 Solve a linear equation with a single unknown where obtaining the solution requires two or more steps (no brackets)

2.2.4 Solve efficiently a linear equation with a single unknown involving brackets

We have produced guidance documents that offer an overview of each core concept, as well as an overview of the content of each statement of knowledge, skills and understanding. We have also broken down each of the latter into a series of key ideas to support planning, with some of the key ideas exemplified as to what teaching for mastery may look like.

We make no suggestion that each key idea represents a lesson. Rather, the fine-grained distinctions we offer in these key ideas are intended to help you think about the learning journey irrespective of the number of lessons taught.

Not all key ideas are of equal weight and the amount of classroom time required for them to be mastered will vary, but each step is a noteworthy contribution to the statement of knowledge, skills and understanding with which it is associated.

These materials are designed for teachers to use collaboratively when planning how they will teach for a secure and deep understanding of mathematics throughout Key Stage 3. They are underpinned by a clear set of pedagogical principles and practices.

The *Operating on number* core concept guidance documents[1] can be downloaded from the NCETM website.

Links to the national curriculum

A mapping[2] of all statements of knowledge skills and understanding to the national curriculum Key Stage 3 programme of study is available on the NCETM website.

Previous learning

From Upper Key Stage 2, students will bring experience of:

- multiplying multi-digit numbers up to four digits by a two-digit whole number using the formal written method of long multiplication
- dividing numbers up to four digits by a two-digit whole number using the formal written method of long division, and interpreting remainders as whole number remainders, fractions, or by rounding, as appropriate for the context
- dividing numbers up to four digits by a two-digit number using the formal written method of short division where appropriate, and interpreting remainders according to the context
- performing mental calculations, including with mixed operations and large numbers
- using their knowledge of the order of operations to carry out calculations involving the four operations
- solving addition and subtraction multi-step problems in contexts, deciding which operations and methods to use and why
- solving problems involving addition, subtraction, multiplication and division
- using estimation to check answers to calculations and determine, in the context of a problem, an appropriate degree of accuracy
- adding and subtracting fractions with different denominators and mixed numbers, using the concept of equivalent fractions
- multiplying simple pairs of proper fractions, writing the answer in its simplest form [e.g. $\frac{1}{4}\times\frac{1}{2}=\frac{1}{8}$]
- dividing proper fractions by whole numbers [e.g. $\frac{1}{3}\div 2=\frac{1}{6}$]
- multiplying one-digit numbers with up to two decimal places by whole numbers
- using written division methods in cases where the answer has up to two decimal places.

Future learning

In Key Stage 4, students will build on the core concepts in this mathematical theme to:

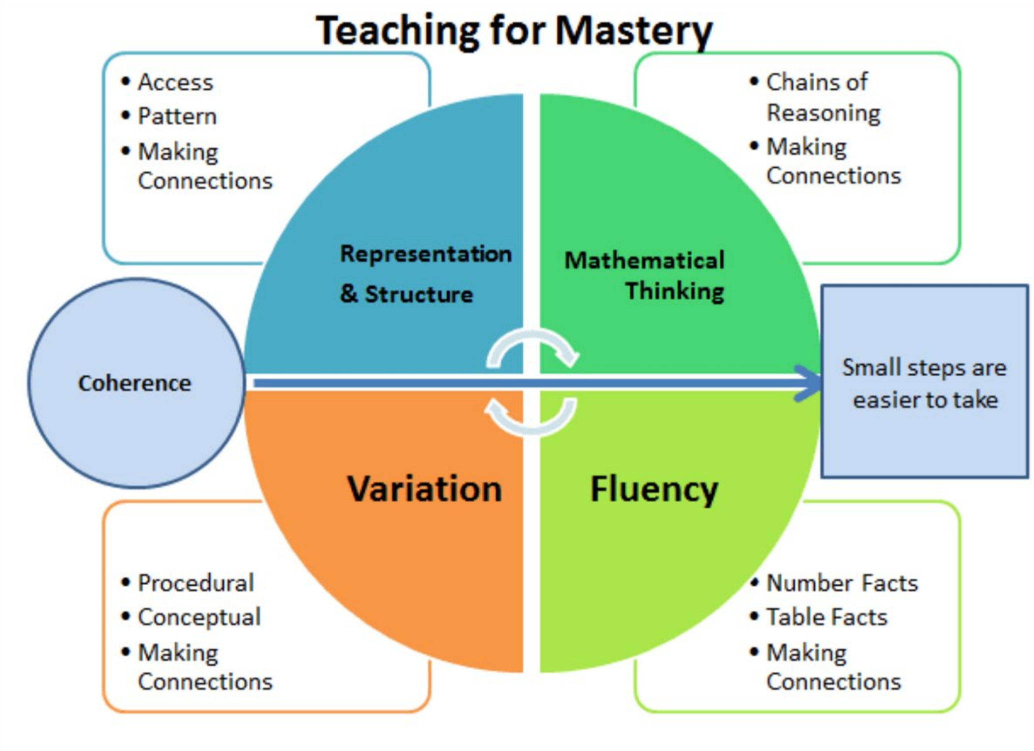
- calculate with roots and with integer {and fractional} indices
- calculate exactly with fractions, {surds} and multiples of π; {simplify surd expressions involving squares [e.g. $\sqrt{12}=\sqrt{(4\times 3)}=\sqrt{4}\times\sqrt{3}=2\sqrt{3}$] and rationalise denominators}
- calculate with numbers in standard form $A \times 10^n$, where $1 \le A < 10$ and n is an integer
- solve two simultaneous equations in two variables (linear/linear {or linear/quadratic}) algebraically; find approximate solutions using a graph
- translate simple situations or procedures into algebraic expressions or formulae; derive an equation (or two simultaneous equations), solve the equation(s) and interpret the solution
- solve linear inequalities in one {or two} variable{s}, {and quadratic inequalities in one variable}; represent the solution set on a number line, {using set notation and on a graph}.

Please note: Braces { } indicate additional mathematical content to be taught to more highly attaining students. Square brackets [ ] indicate content schools are not required to teach by law.

Teaching for mastery

A central component in the NCETM/Maths Hubs programmes to support the development of teaching for mastery has been discussion of Five Big Ideas[3] underpinning teaching for mastery. These are:

- Coherence
- Representation and structure
- Variation
- Fluency
- Mathematical thinking



The sections below offer guidance about how these ideas relate to *Operating on number*.

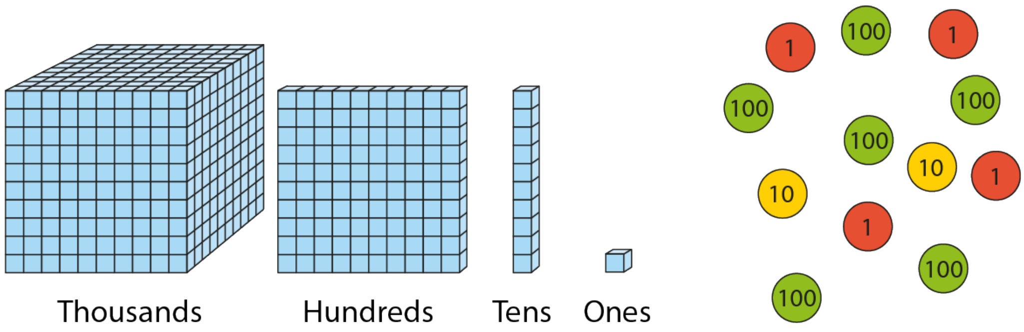
Coherence

It is important to find a balance between focusing on important elements of this theme where it is useful to plan a coherent set of small steps (for example, when multiplying fractions, what ideas to introduce first, what should come next, and so on) and appreciating how each idea is connected to others in the theme. For example, when thinking about multiplying two fractions it will be important to refer to multiplication as scaling and to use the area model of multiplication (both previously encountered with integers) to see this as one concept applied to all numbers. You should also relate fractions to decimals so that students see them as connected ideas.

It will also be important to use and apply previously understood ideas and techniques in subsequent work wherever possible. For example, when introducing linear equations, make sure that fractions and decimals are used.

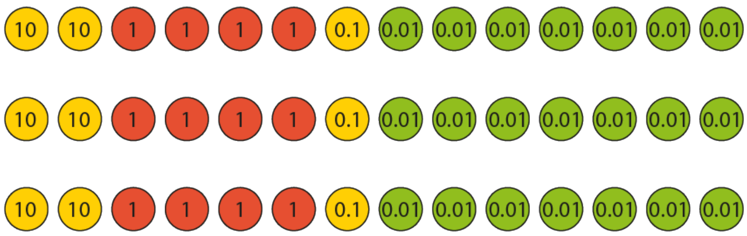
Representation and structure

<table>
<tr><th>Representations</th><th>Structural understanding</th></tr>
<tr><td>Dienes base-ten blocks

Place-value counters</td><td>


The use of Dienes blocks and, particularly, place-value counters can support students’ understanding of certain algorithms and procedures when they are used alongside the more symbolic representations of the calculations. For example, 24.17 can be modelled using place-value counters as:

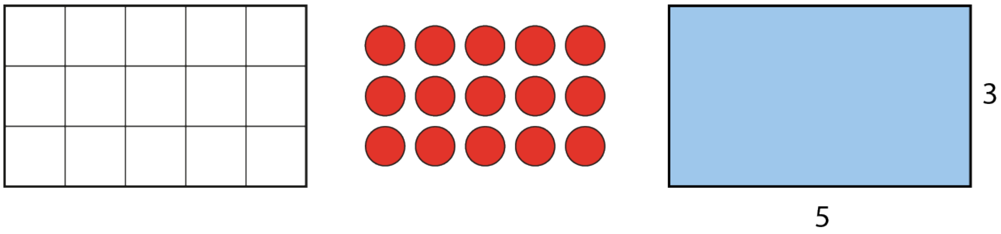
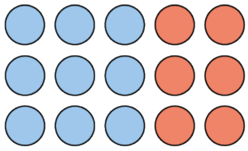
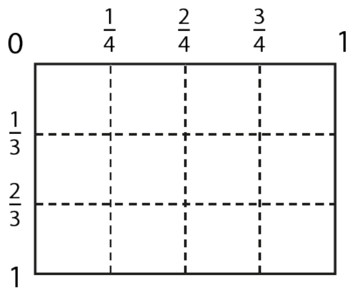
By arranging three rows of these alongside the written calculation:

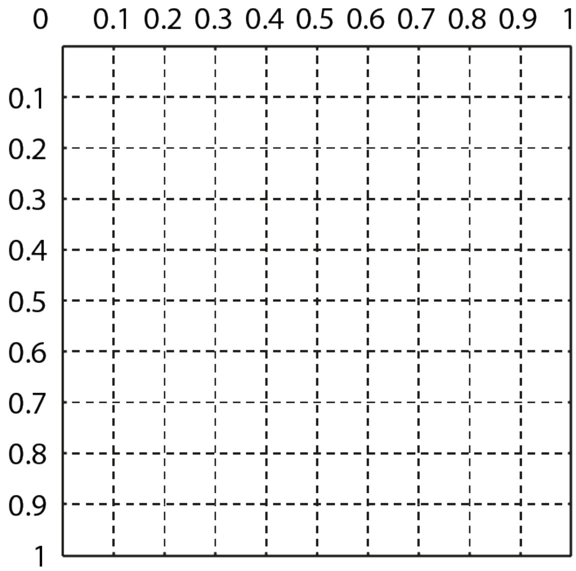
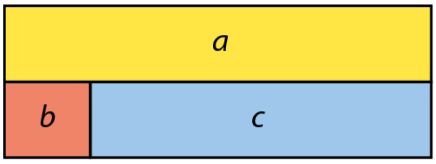
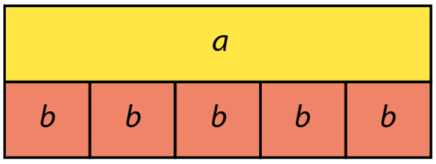
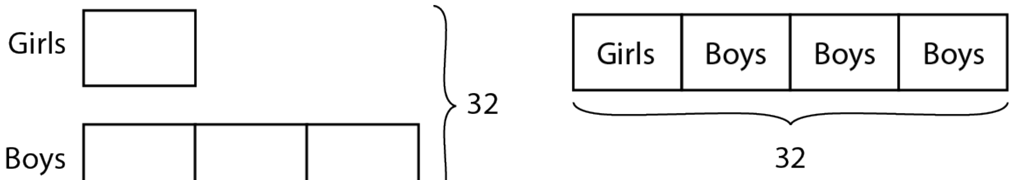
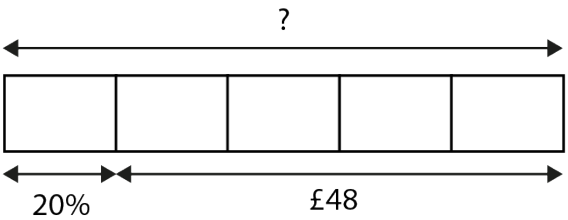


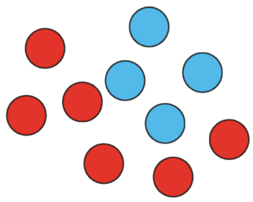
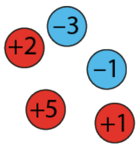
$$\begin{array}{r} 24.17 \\ \times \quad 3 \\ \hline 72.51 \\ \hline \end{array}$$

students can make sense of the various elements of the written method (for example, why the three 0.07s can be re-grouped to form two 0.1s and one 0.01, resulting in a ‘1’ in the hundredths column and a ‘2’ in the tenths column, etc.).
</td></tr>
<tr><td>Gattegno chart</td><td>
<table>
<tr><td>1 000</td><td>2 000</td><td>3 000</td><td>4 000</td><td>5 000</td><td>6 000</td><td>7 000</td><td>8 000</td><td>9 000</td></tr>
<tr><td>100</td><td>200</td><td>300</td><td>400</td><td>500</td><td>600</td><td>700</td><td>800</td><td>900</td></tr>
<tr><td>10</td><td>20</td><td>30</td><td>40</td><td>50</td><td>60</td><td>70</td><td>80</td><td>90</td></tr>
<tr><td>1</td><td>2</td><td>3</td><td>4</td><td>5</td><td>6</td><td>7</td><td>8</td><td>9</td></tr>
<tr><td>0.1</td><td>0.2</td><td>0.3</td><td>0.4</td><td>0.5</td><td>0.6</td><td>0.7</td><td>0.8</td><td>0.9</td></tr>
<tr><td>0.01</td><td>0.02</td><td>0.03</td><td>0.04</td><td>0.05</td><td>0.06</td><td>0.07</td><td>0.08</td><td>0.09</td></tr>
</table>
Moving (up or down) from one row to another to represent multiplying and dividing by powers of ten helps students to understand these operations and their results.

The NRICH article ‘Activities on the Gattegno Chart’[4] by Alf Coles provides further details of how the Gattegno chart might be used.
</td></tr>
</table>

Arrays and other area models for multiplication	 The array is a useful image for revealing the commutative (i.e. $3 \times 5 = 5 \times 3$) and distributive (i.e. $3 \times (2 + 3) = 3 \times 2 + 3 \times 3$) properties of multiplication, as represented below:  When the rectangle has continuous measures for the dimensions, it becomes a useful way of thinking about the product of any two numbers including decimals and fractions. For example:  The diagram above helps students to make sense of the various calculations: $\frac{1}{4} \times \frac{1}{3}$, $\frac{1}{4} \times \frac{2}{3}$, $\frac{3}{4} \times \frac{2}{3}$, etc. Similarly, the diagram below supports understanding of why 0.1 multiplied by 0.1 gives 0.01 and how the product 3×7, for example, is connected to 0.3×0.7:

<table>
<tr><td></td><td></td></tr>
<tr><td>Bar models</td><td>

Bar models can be very useful to support students in representing (literally, re-presenting) problems to reveal additive and multiplicative structures (as appropriate) and thus become aware of how to proceed to a solution. For example:
‘There are 32 students in the class.
There are three times as many boys as girls. How many girls are there?’

‘A computer game was reduced in a sale by 20% and it now costs £48.
What was the original price?’
</td></tr>
</table>

Double-sided (positive and negative) counters	Counters with different colours can be used to represent positive and negative quantities, and a collection of such counters can help to give meaning to the combining of such quantities:  For example, this representation (with red representing positive and blue representing negative) depicts the sum of +6 (positive 6) and –4 (negative 4), i.e. +6 + –4. By pairing positives and negatives to give zero (sometimes referred to as 'zero pairs'), it can be seen that +6 + –4 = +2. By adding and subtracting (i.e. removing) both positive and negative quantities from such collections, students can make sense of the addition and subtraction of directed numbers. The counters can be left blank initially, with each counter being positive 1 or negative 1. Later on, symbols can be added to enable students to gain fluency with the abstract notation of directed numbers. 

Further guidance on using representations[5] in Key Stage 3 is available on the NCETM website.

Variation

Three aspects of variation that can be usefully employed:

1. Careful **choice of exercises** to 'home in' on the important concept. For example, when working on division by a fraction, choose examples which draw attention to the fact that, as the denominator of the divisor increases, the quotient increases and as the numerator of the divisor increases, the quotient decreases, e.g. $10 \div \frac{1}{2}$; $10 \div \frac{1}{3}$; $10 \div \frac{1}{5}$; $10 \div \frac{2}{5}$; $10 \div \frac{3}{5}$, etc.
2. Careful **choice of examples** to include *'what it is'* (using non-standard as well as standard examples) and *'what it is not'*. For example, when offering examples (or when inviting students to offer examples) of linear equations, include examples of the form:
   a) $a + bx = c$ or $c = a + bx$, where a and/or b might be negative, decimal or fraction (non-standard)
   b) $x^2 + 3 = 7$ or $x + \frac{4}{x} = 5$ (what it is not)

   as well as the standard $ax + b = c$ (where $a > 0$; $c > b > 0$).

3. Rather than focusing on the answer and asking only that students solve a problem, inviting students to see **in how many different ways they can solve a problem** can prompt important discussions about methods and processes, and support students' development of increasingly efficient, creative and elegant approaches. For example, solving 25×28 in different ways will enable discussions about the commutative, distributive and associative laws. This will support students' development of increasingly efficient, creative and elegant approaches.

   Similarly, discussing whether to expand brackets as a first step when solving an equation, such as $6(2x + 1) = 12$ or $6(2x + 1) = 13$, will encourage students to work flexibly with mathematical structure, rather than repeating a learnt routine.

Fluency

A key aspect of fluency is the ability to choose the most efficient strategy for a problem or calculation; to know when a standard method is appropriate and when alternative methods might be more efficient. For example, solving the equation $3(2x – 5) = 21$ with or without multiplying out the brackets, or calculating $5\,004 – 4\,995$ quickly by thinking of subtraction as difference. A key element of *Operating on number* is having a deep and secure understanding of the structures underpinning a wide range of calculation methods so that intelligent choices can be made.

Mathematical thinking

Throughout all the work that falls within *Operating on number*, the emphasis is on understanding the rules of arithmetic on which various calculation strategies are based. It is vital that students are prompted to reason, explain, conjecture and prove through carefully planned teacher–student and student–student discussion, and not merely to listen to and follow carefully constructed teacher demonstrations and explanations. For example, when learning about the structures underpinning multiplication and division, the following questions could be discussed and explored:

- *'In addition, a number can be added to one of the addends and subtracted from the other and the answer stays the same, e.g.* $64 + 29 = 63 + 30$*. Why does this not work for multiplication? – i.e.* $64 \times 29 \neq 63 \times 30$*?'*
- *'Which is bigger,* 64×29 *or* 63×30*? How much bigger is it? Can you draw a diagram to show why?'*

Further reading

NCETM secondary assessment materials[6]
Exemplar questions, tasks and activities, which may be used to support teaching and assessment. The assessment materials are mapped against the key mathematical skills and concepts within the national curriculum Key Stage 3 programme of study. Of particular relevance to *Operating on number* are the sections focusing on: arithmetic procedures (pages 10–12); the use of a calculator (page 13) and solving equations (pages 23–24).

Weblinks

1 Theme 2: *Operating on number* – core concept guidance documents
https://www.ncetm.org.uk/resources/53531

2 NCETM Key Stage 3 mastery curriculum structure, including national curriculum mapping
https://www.ncetm.org.uk/secondarymasterypd#curriculum_structure

3 Five Big Ideas in Teaching for Mastery
https://www.ncetm.org.uk/resources/50042

[4] NRICH: 'Activities on the Gattegno Chart' article
https://nrich.maths.org/10741

[5] Representations in Key Stage 3 – guidance documents
https://www.ncetm.org.uk/resources/53609

[6] NCETM secondary assessment materials
https://www.ncetm.org.uk/resources/51246